MAT 167 TEST 2 FORM B (DERIVATIVES)

Directions: Solve each required problem **on a separate sheet of paper. Use pencil and show all work;** I deduct points for using pen or skipping important steps. **You must shut off your cell phone.** Some problems are worth more than others. Take your time; **quality is preferred to quantity.** I encourage you to ask questions.

1. (a) How would you describe a differentiable function, in geometric terms?
   Solution: Smooth. (That one word would suffice, but if you want to add "no corners, cusps, or discontinuities," I'm happy with that, too. Or with that, instead.)
   (b) If a function is continuous at a point a, is it differentiable at a?
   Solution: Not necessarily, as $f(x) = |x|$ shows. I accepted "no", though technically "no" is incorrect: it isn't **always** ***false.***
   (c) If a function is differentiable at a point a, is it continuous at a?
   Solution: Yes. (That one word would suffice. If you want to say more, you should say that, "A function cannot be smooth if it has corners, cusps, and discontinuities.)

2. Let $f(x) = 13$.
   (a) Give an *intuitive* explanation as to why $f'(x) = 0$ at every value of x.
   Solution: The function is a horizontal line, so its slope is 0. To go in the same direction as the "curve", the line tangent to f must have the same slope. The slope of the tangent line is the derivative, so the derivative of f is 0.
   (b) Use the *precise definition* of the derivative to show that $f'(x) = 0$ at every value of x.
   Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{3-3}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

3. Compute the derivatives of the following functions.

   (a) $f(x) = x - \dfrac{3}{x^2}$ (b) $g(x) = \cos(2x) - 3\sec x$ (c) $h(x) = xe^x$

   (d) $F(x) = x^x$ (e) $G(x) = \dfrac{e^x (x-3)^{14} (2x+1)^3}{x^5 (x-3)^2 \cos^3 x}$ (f) $H(x) = \left(1 + \tan^{-1} x\right)^2$

Solutions:

(a) I personally would rewrite $f(x) = x - 3x^{-2}$*, so that it is "easy" to see that* $f'(x) = 1 - 3(-2)x^{-3}$*, or* $f'(x) = 1 + \frac{6}{x^3}$*. You could also use the quotient rule, but try to remember that the derivative of a constant is 0:*

$$f'(x) = 1 - \frac{0 \cdot x^2 - 3 \cdot 2x^1}{\left(x^2\right)^2} = 1 + \frac{6x}{x^4} = 1 + \frac{6}{x^3}.$$

(b) Notice the chain rule in action, as well as the constant multiple rule: $g'(x) = -\sin(2x) \cdot 2 - 3\sec x \tan x$.

(c) Notice the product rule in action: $h'(x) = 1 \cdot e^x + x \cdot e^x = e^x + xe^x$*. I'm okay with that answer, but if you have a yen to do more, the only* correct *option you have is to factor:* $h'(x) = e^x(1+x)$.

(d) The only *way to do this problem is via logarithmic differentiation:*

$$\begin{aligned}
y &= x^x \\
\ln y &= \ln x^x \\
\ln y &= x \ln x && \text{(power rule for logarithms)} \\
\frac{1}{y} \cdot y' &= 1 \cdot \ln x + x \cdot \frac{1}{x} && \text{(product rule of differentiation)} \\
\frac{y'}{y} &= \ln x + 1 && \text{(a little algebra makes the world a prettier place)} \\
y' &= y(\ln x + 1) && \text{(isolate } y'\text{)} \\
y' &= x^x(\ln x + 1) && \text{(substitute original value of } y\text{)}
\end{aligned}$$

(e) The smart *way to do this is by logarithmic differentiation:*

$$\begin{aligned}
\ln y &= \ln \frac{e^x (x-3)^{14} (2x+1)^3}{x^5 (x-3)^2 \cos^3 x} && \textit{(}G(x)\textit{ is a fancy name for } y\textit{)} \\
&= \ln\left[e^x (x-3)^{14} (2x+1)^3\right] && \textit{(quotient rule for logarithms)} \\
&\quad - \ln\left[x^5 (x-3)^2 \cos^3 x\right] \\
&= \ln e^x + \ln (x-3)^{14} + \ln (2x+1)^3 && \textit{(product rule for logarithms)} \\
&\quad - \left[\ln x^5 + \ln (x-3)^2 + \ln \cos^3 x\right] \\
&= x + 14\ln(x-3) + 3\ln(2x+1) && \textit{(}e^x \textit{ and } \ln x \textit{ are inverses,} \\
&\quad - [5\ln x + 2\ln(x-3) + 3\ln\cos x] && \textit{and power rule for logarithms)} \\
\frac{1}{y} \cdot y' &= 1 + 14 \cdot \frac{1}{x-3} \cdot 1 + 3 \cdot \frac{1}{2x+1} \cdot 2 && \textit{(differentiate, using constant multiple} \\
&\quad - 5 \cdot \frac{1}{x} - 2 \cdot \frac{1}{x-3} \cdot 1 - 3 \cdot \frac{1}{\cos x} \cdot (-\sin x) && \textit{and chain rules)}
\end{aligned}$$

$$\frac{y'}{y} = 1 + \frac{14}{x-3} + \frac{6}{2x+1} - \frac{5}{x} - \frac{2}{x-3} + 3\tan x \qquad \textit{(a little algebraic art)}$$

$$y' = y\left(1 + \frac{14}{x-3} + \frac{6}{2x+1} - \frac{5}{x} - \frac{2}{x-3} + 3\tan x\right) \textit{(isolate } y\textit{)}$$

$$y' = \frac{e^x (x-3)^{14} (2x+1)^3}{x^5 (x-3)^2 \cos^3 x} \qquad \textit{(substitute original value of } y\textit{)}$$

$$\cdot \left(1 + \frac{14}{x-3} + \frac{6}{2x+1} - \frac{5}{x} - \frac{2}{x-3} + 3\tan x\right)$$

If you're really uncomfortable with logarithms, you can do it with the quotient, product, and chain rules for derivatives, but it's nasty as all get-out:

$$G'(x) = \frac{\left[e^x (x-3)^{14} (2x+1)^3 + e^x \cdot 14 (x-3)^{13} \cdot 1 \cdot (2x+1)^3 + e^x (x-3)^{14} \cdot 3 (2x+1)^2 \cdot 2\right] \cdot x^5 (x-3)^2 \cos^3 x \cdots}{\left[x^5 (x-3)^2 \cos^3 x\right]^2}$$

$$\frac{e^x (x-3)^{14} (2x+1)^3 \left[5x^4 (x-3)^2 \cos^3 x + x^5 \cdot 2(x-3) \cdot \cos^3 x + x^5 (x-3)^2 \cdot 3\cos^2 x \cdot (-\sin x)\right]}{},$$

and no, I **won't** ***simplify that further, thank you.***

(d) Notice the chain rule in action, and the fact that $\tan^{-1} x$ ***has nothing to do with*** $\sec x$***:***

$$F'(x) = 2\left(1 + \tan^{-1} x\right) \cdot \left(0 + \frac{1}{1+x^2}\right) = \frac{2\left(1 + \tan^{-1} x\right)}{1+x^2}.$$

4. Use implicit differentiation to compute the equation of the line tangent to the curve described by $y^3 + y = x^3 - x$ at the point $(-1, 0)$.

   Solution:

$$3y^2 \cdot y' + y' = 3x^2 - 1 \textit{ (differentiate — don't forget } y'\textit{)}$$

$$\left(3y^2 + 1\right) y' = 3x^2 - 1 \textit{ (factor the common term)}$$

$$y' = \frac{3x^2 - 1}{3y^2 + 1} \textit{(isolate } y'\textit{)}$$

We can now use the derivative to find the slope of the tangent line, by substituting the given values of x ***and*** y***:***

$$m_{tan} = \frac{3 \cdot (-1)^2 - 1}{3 \cdot 0^2 + 1} = \frac{2}{1} = 2.$$

This allows us to write the equation of a line:

$$y - 0 = 2 \cdot (x - (-1)).$$

If you had written only this far, I would be happy. After all, **it is the equation of a line!** ***However, if you decided to take it farther, then be careful. You should have this:***

$$y = 2x + 2.$$

5. Use the graph of $f(x)$ given below to answer the following questions.
   (a) At what point(s) is f discontinuous?
   Solution: Never! There are no holes, jumps, or asymptotes.
   (b) At what point(s) is f nondifferentiable?
   Solution: At $(1,-1)$.
   (c) At what point(s) does f have the *largest slope?*
   Solution: Any point on the interval $\left(1,3\frac{1}{2}\right)$ ***would make me happy.***
   (d) Rank the values $x=1/2$, $x=2$, $x=3\frac{1}{2}$, and $x=4\frac{1}{2}$ from smallest to largest slope of f.
   Solution: Negative slopes are smaller, so in order, they should be:

$$\frac{1}{2},\ 4\frac{1}{2},\ 3\frac{1}{2},\ 2.$$