MAT 167 TEST 2 FORM A (DERIVATIVES)

Directions: Solve each required problem **on a separate sheet of paper. Use pencil and show all work;** I deduct points for using pen or skipping important steps. **You must shut off your cell phone.** Some problems are worth more than others. Take your time; **quality is preferred to quantity.** I encourage you to ask questions.

1. Let $f(x)$ be a function that is defined at $x = a$. Give *intuitive* definitions of:
   (a) a line tangent to $f(x)$ at a;
   Solution: A line that passes through the graph of f at a and has the same direction as f at that point.
   (b) the derivative of $f(x)$ at x.
   Solution: The slope of the tangent line.

2. Let $f(x) = 3x$.
   (a) Give an *intuitive* explanation as to why $f'(x) = 3$ at every value of x.
   Solution: Since f is a line with slope 3, the tangent line cannot go in the same direction unless it also has slope 3. The slope of the tangent line is the derivative, so the derivative of f is 3.
   (b) Use the *precise definition* of the derivative to show that $f'(x) = 3$ at every value of x.
   Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{3(x+h)-3x}{h} \\
&= \lim_{h\to 0} \frac{\cancel{3x}+3h-\cancel{3x}}{h} \\
&= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}} \\
&= \lim_{h\to 0} 3 \\
&= 3.
\end{aligned}$$

3. Compute the derivatives of the following functions.

(a) $f(x) = x^2 - \dfrac{3}{x}$ (b) $g(x) = \sin(2x) - 3\tan x$ (c) $h(x) = xe^x$

(d) $F(x) = \left(1+\sin^{-1} x\right)^2$ (e) $G(x) = \dfrac{e^x (x-3)^{14} (2x+1)^3}{x^5 (x-3)^2 \cos^3 x}$ (f) $H(x) = x^x$

Solutions:

(a) I personally would rewrite $f(x) = x^2 - 3x^{-1}$***, so that it is "easy" to see that*** $f'(x) = 2x^1 - 3(-1)x^{-2}$***, or*** $f'(x) = 2x + \frac{3}{x^2}$***. You could also use the quotient rule, but try to remember that the derivative of a constant is*** 0***:***

$$f'(x) = 2x^1 - \frac{0 \cdot x - 3 \cdot 1}{x^2} = 2x + \frac{3}{x^2}.$$

(b) Notice the chain rule in action, as well as the constant multiple rule: $g'(x) = \cos(2x) \cdot 2 - 3\sec^2 x$.

(c) Notice the product rule in action: $h'(x) = 1 \cdot e^x + x \cdot e^x = e^x + xe^x$***. I'm okay with that answer, but if you have a yen to do more, the only*** correct ***option you have is to factor:*** $h'(x) = e^x(1+x)$.

(d) Notice the chain rule in action, and the fact that $\sin^{-1} x$ ***is not the same as*** $\csc x$***:***

$$F'(x) = 2\left(1 + \sin^{-1} x\right) \cdot \left(0 + \frac{1}{\sqrt{1-x^2}}\right) = \frac{2\left(1+\sin^{-1} x\right)}{\sqrt{1-x^2}}.$$

(e) The smart ***way to do this is by logarithmic differentiation:***

$$\begin{aligned}
\ln y &= \ln \frac{e^x (x-3)^{14} (2x+1)^3}{x^5 (x-3)^2 \cos^3 x} && \textit{(}G(x)\textit{ is a fancy name for }y\textit{)}\\
&= \ln\left[e^x (x-3)^{14} (2x+1)^3\right] && \textit{(quotient rule for logarithms)}\\
&\quad - \ln\left[x^5 (x-3)^2 \cos^3 x\right] \\
&= \ln e^x + \ln (x-3)^{14} + \ln (2x+1)^3 && \textit{(product rule for logarithms)}\\
&\quad - \left[\ln x^5 + \ln (x-3)^2 + \ln \cos^3 x\right] \\
&= x + 14\ln(x-3) + 3\ln(2x+1) && \textit{(}e^x\textit{ and }\ln x\textit{ are inverses,}\\
&\quad - \left[5\ln x + 2\ln(x-3) + 3\ln\cos x\right] && \quad\textit{and power rule for logarithms)}\\
\frac{1}{y} \cdot y' &= 1 + 14 \cdot \frac{1}{x-3} \cdot 1 + 3 \cdot \frac{1}{2x+1} \cdot 2 && \textit{(differentiate, using constant multiple}\\
&\quad - 5 \cdot \frac{1}{x} - 2 \cdot \frac{1}{x-3} \cdot 1 - 3 \cdot \frac{1}{\cos x} \cdot (-\sin x) && \quad\textit{and chain rules)}\\
\frac{y'}{y} &= 1 + \frac{14}{x-3} + \frac{6}{2x+1} - \frac{5}{x} - \frac{2}{x-3} + 3\tan x && \textit{(a little algebraic art)}\\
y' &= y\left(1 + \frac{14}{x-3} + \frac{6}{2x+1} - \frac{5}{x} - \frac{2}{x-3} + 3\tan x\right) && \textit{(isolate }y\textit{)}\\
y' &= \frac{e^x (x-3)^{14} (2x+1)^3}{x^5 (x-3)^2 \cos^3 x} && \textit{(substitute original value of }y\textit{)}\\
&\quad \cdot \left(1 + \frac{14}{x-3} + \frac{6}{2x+1} - \frac{5}{x} - \frac{2}{x-3} + 3\tan x\right)
\end{aligned}$$

If you're really uncomfortable with logarithms, you can do it with the quotient, product, and chain rules for derivatives, but it's nasty as all get-out:

$$G'(x) = \frac{\left[e^x (x-3)^{14} (2x+1)^3 + e^x \cdot 14 (x-3)^{13} \cdot 1 \cdot (2x+1)^3 + e^x (x-3)^{14} \cdot 3 (2x+1)^2 \cdot 2\right] \cdot x^5 (x-3)^2 \cos^3 x \cdots}{\left[x^5 (x-3)^2 \cos^3 x\right]^2}$$

$$\frac{e^x (x-3)^{14} (2x+1)^3 \left[5x^4 (x-3)^2 \cos^3 x + x^5 \cdot 2 (x-3) \cdot \cos^3 x + x^5 (x-3)^2 \cdot 3 \cos^2 x \cdot (-\sin x)\right]}{},$$

and no, I **won't** *simplify that further, thank you.*

(f) The **only** *way to do this problem is via logarithmic differentiation:*

$$
\begin{aligned}
y &= x^x \\
\ln y &= \ln x^x \\
\ln y &= x \ln x && \text{(power rule for logarithms)} \\
\frac{1}{y} \cdot y' &= 1 \cdot \ln x + x \cdot \frac{1}{x} && \text{(product rule of differentiation)} \\
\frac{y'}{y} &= \ln x + 1 && \text{(a little algebra makes the world a prettier place)} \\
y' &= y (\ln x + 1) && \text{(isolate } y'\text{)} \\
y' &= x^x (\ln x + 1) && \text{(substitute original value of } y\text{)}
\end{aligned}
$$

4. Use implicit differentiation to compute the equation of the line tangent to the curve described by $y^3 + y = x^3 - x$ at the point $(1, 0)$.

   Solution:

$$
\begin{aligned}
3y^2 \cdot y' + y' &= 3x^2 - 1 \textit{ (differentiate — don't forget } y'\textit{)} \\
\left(3y^2 + 1\right) y' &= 3x^2 - 1 \textit{ (factor the common term)} \\
y' &= \frac{3x^2 - 1}{3y^2 + 1} \textit{(isolate } y'\textit{)}
\end{aligned}
$$

   We can now use the derivative to find the slope of the tangent line, by substituting the given values of x and y:

$$m_{tan} = \frac{3 \cdot 1^2 - 1}{3 \cdot 0^2 + 1} = \frac{2}{1} = 2.$$

   This allows us to write the equation of a line:

$$y - 0 = 2 \cdot (x - 1).$$

   If you had written only this far, I would be happy. After all, **it is the equation of a line!** *However, if you decided to take it farther, then be careful. You should have this:*

$$y = 2x - 2.$$

5. Use the graph of $f(x)$ given below to answer the following questions.

(a) At what point(s) is f discontinuous?
Solution: Never! There are no holes, jumps, or asymptotes.
(b) At what point(s) is f nondifferentiable?
Solution: At $(1,1)$ ***and*** $(3,-1)$.
(c) At what point(s) does f have the *largest slope*?
Solution: Technically, never. However, if you wrote something close to $x = 1$ ***and/or*** $x = 3$, ***I let it slide.***
(d) Rank the values $x = 1/2$, $x = 2$, $x = 3\frac{1}{2}$, and $x = 4\frac{1}{2}$ from smallest to largest slope of f.
Solution: Negative slopes are smaller, so in order, they should be:

$$2, \frac{1}{2}, 4\frac{1}{2}, 3\frac{1}{2}.$$

I could see switching $\frac{1}{2}$ ***and*** $4\frac{1}{2}$, ***but not any of the others.***