MAT 167 TEST 3 REVIEW

1.1. Provide an "intuitive," "geometric," and/or "precise" descriptions for *four* of the concepts or results listed below, as directed.

(a) The **Mean Value Theorem**. *(intuitive/geometric and precise)*
(b) A **critical point**. *(intuitive/geometric and precise)*
(c) **Concave up** or **down**. *(intuitive/geometric and precise)*
(d) An **inflection point**. *(intuitive/geometric and precise)*
(e) A **local maximum** or **minimum**.

Solutions:

(a) *intuitive:* If a function f is very well-behaved on the interval $[a,b]$, then we can find a point between a and b whose tangent line has the same slope as the secant line connecting a's and b's points.
precise: If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

(b) *intuitive:* A critical point is where a graph is either level or not smooth.
precise: $x = a$ is a critical point if $f(a) = 0$ or $f'(a)$ does not exist.

(c) *intuitive:* A function is concave up (resp. down) when its curve bends upward (resp. downward).
precise: A function is concave up (resp. down) at $x = a$ when its derivative is increasing (resp. decreasing) at $x = a$; or, when $f''(a) > 0$ (resp. $f''(a) < 0$).

(d) *intuitive **and** precise:* An inflection point is where a function's concavity changes from up to down or vice versa.

(e) *intuitive:* A local maximum (resp. minimum) is the largest (resp. smallest) y-value in a neighborhood.
precise: $x = c$ is a local maximum (resp. minimum) if $c \in (a,b)$ and $f(c) \geq f(x)$ for all $x \in (a,b)$.

1.2. Compute the derivatives of the following functions or equations. (Mostly for review, but one involves new material you can expect to see on the test.)

(a) $g(x) = \dfrac{\sin(x)+1}{\cos x}$ (b) $f(t) = \arcsin(3t)$

(c) $y = 10\ln\left(x^2+1\right)$ (d) $p(x) = e^{2x}\cos 2x$

(e) $\sin(x+y) = \sin x + \sin y$ (f) $H(x) = (3x)^{3x}$

Solutions:

(a) Use the quotient rule:

$$\begin{aligned} g'(x) &= \frac{\left[\frac{d}{dx}(\sin x+1)\right]\cdot\cos x - (\sin x+1)\cdot\frac{d}{dx}\cos x}{\cos^2 x} \\ &= \frac{\cos x\cdot\cos x - (\sin x+1)\cdot(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin x(\sin x+1)}{\cos^2 x}. \end{aligned}$$

You can simplify this to

$$g'(x) = \frac{1+\sin x}{\cos^2 x}$$

and possibly to other expressions, but the second line would actually suffice.

(b) This involves one of those functions you see often enough to forget. Also, you'll need the Chain Rule:

$$\frac{d}{dt}\arcsin(3t) = \frac{1}{1+(3t)^2}\cdot 3 = \frac{3}{1+9t^2}.$$

(c) Again, you need the Chain Rule:

$$\frac{dy}{dx} = \underbrace{10}_{\text{const. mult.}}\cdot\underbrace{\frac{1}{x^2+1}}_{\text{deriv. of log}}\cdot\underbrace{2x}_{\text{chain}} = \frac{20x}{x^2+1}.$$

(d) You'll want the product rule and the Chian Rule:

$$p'(x) = \left(e^{2x}\cdot 2\right)\cdot\cos 2x + e^{2x}\cdot(-\sin 2x\cdot 2) = 2e^{2x}(\cos 2x - \sin 2x).$$

(e) Here you need Implicit Differentiation, which *is* on Test 3!

$$\begin{aligned}
\sin(x+y) &= \sin x + \sin y \\
\cos(x+y)\cdot\left(1+y'\right) &= \cos x + \cos y \cdot y' \\
\cos(x+y) + y'\cos(x+y) &= \cos x + y'\cos y \\
y'\cos(x+y) - y'\cos y &= \cos x - \cos(x+y) \\
y'\left[\cos(x+y) - \cos y\right] &= \cos x - \cos(x+y) \\
y' &= \frac{\cos x - \cos(x+y)}{\cos(x+y) - \cos y}\,.
\end{aligned}$$

(f) Here you need Logarithmic Differentiation, which is not on Test 3, but you'll still need it for the final! Let $y = (3x)^{3x}$. Then

$$\begin{aligned}
\ln y &= \ln\left((3x)^{3x}\right) \\
\ln y &= 3x \cdot \ln(3x) \\
\frac{d}{dx}\left[\ln y \right.&= \left. 3x \cdot \ln(3x)\right] \\
\frac{1}{y}\cdot y' &= \underbrace{3\cdot\ln(3x) + 3x\cdot\frac{1}{3x}\cdot\underbrace{3}_{\text{chain}}}_{\text{product rule}} \\
y' &= y\left[3\ln(3x) + 3\right] \\
y' &= (3x)^{3x}\left[3\ln(3x) + 3\right]\,.
\end{aligned}$$

1.4. In the study of ecosystems, predator-prey models are often used to study the interaction between species. Consider populations of hungry hawks, given by $H(t)$, and scrumptious squirrels, given by $S(t)$, where we measure t in years. The interaction can be modeled by the equations

$$\frac{dH}{dt} = -aH + bHS \quad \text{and} \quad \frac{dS}{dt} = cS - dHS.$$

(a) What values of dH/dt and dS/dt correspond to stable populations?

Suppose $a = .1$, $b = .0002$, $c = .2$, $d = .02$.

(b) If $H = 20$ and $S = 3400$, how does each population change: do they increase or decrease?
(c) Do there exist values for H and S such that both populations live in balance? (That is, both populations are stable.)

Solutions:

(a) "Stable" means the population isn't changing, so $dH/dt = 0$ and $dS/dt = 0$.

(b) By substitution,

$$\frac{dH}{dt} = -0.1 \times 20 + 0.0002 \times 20 \times 3400 = 11.6\,,$$

and

$$\frac{dS}{dt} = 0.2 \times 3400 - 0.02 \times 20 \times 3400 = -680\,.$$

The hungry hawks are increasing by almost 11–12 hawks per year, while the scrumptious squirrels are decreasing by 680 squirrels per year.

(c) We need $dH/dt = 0$ and $dS/dt = 0$ at the same time. That is,

$$0 = -0.1 \times H + 0.0002 \times H \times S \quad \text{and} \quad 0 = 0.2 \times S - 0.02 \times H \times S\,.$$

We can factor a common H from the first equation, yielding

$$0 = H(-0.1 + 0.0002S)\,.$$

By the Zero Product Property, $H = 0$ or $S = 500$. Zero hawks is not really a balance (the squirrels will increase significantly) so we throw out that solution. Similarly, we factor a common S from the second equation, yielding

$$0 = S(0.2 - 0.02H)\,.$$

By the Zero Product Property, $S = 0$ or $H = 10$. Zero squirrels is not really a balance (the hawks will die of starvation) so we throw out that solution. We are left with stability only when we have 10 hungry hawks and 500 scrumptious squirrels.

1.5. Let

$$f(x) = \cos x + \frac{x}{2}.$$

(a) Find the critical point(s) of $f(x)$ on the interval $[0, 2\pi]$.
(*Careful:* This involves a periodic function, so there could be more than one answer. You must list them all.)

(b) Determine the maximum and minimum value(s) of $f(x)$ on the interval $[0, 2\pi]$. Indicate whether each extremum is relative or absolute.

(c) Determine the intervals of increase and intervals of decrease for f on the interval $[0, 2\pi]$.

(d) Find the possible inflection point(s) of $f(x)$ on the interval $[0, 2\pi]$.

(e) Indicate over which subintervals of $[0, 2\pi]$ f is concave up, and over which interval f is concave down.

(f) Identify any horizontal or vertical asymptotes of $f(x)$. If there aren't any, explain why not.

(g) Combine all this information into a sketch of $f(x)$ over the interval $[0, 2\pi]$.

Solutions:

(a) The first derivative is $f'(x) = -\sin x + 1/2$. This is never undefined, so critical points occur only when $f'(x) = 0$; that is, when $\sin x = 1/2$; that is, when $x = \pi/6, 5\pi/6$.

(b) The y-values we have to consider occur at the critical points and at the endpoints:

$$f(0) = \cos 0 + \frac{0}{2} = 1 \qquad \Longrightarrow \text{local min}$$

$$f\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} + \frac{\pi/6}{2} = \frac{\sqrt{3}}{2} + \frac{\pi}{12} \approx 1.13 \qquad \Longrightarrow \text{local max}$$

$$f\left(\frac{5\pi}{6}\right) = \cos\frac{5\pi}{6} + \frac{5\pi/6}{2} = -\frac{\sqrt{3}}{2} + \frac{5\pi}{12} \approx 0.4 \Longrightarrow \text{global min}$$

$$f(2\pi) = \cos(2\pi) + \frac{2\pi}{2} = 1 + \pi \approx 4.1 \qquad \Longrightarrow \text{global max}$$

(c) To find the intervals of increase and decrease, we choose values between each endpoint and/or critical point, and substitute into the derivative:

$f'(x)$	+	+	0	-	0	+	+
x	0	0.1	$\frac{\pi}{6}$	1	$\frac{5\pi}{6}$	π	2π

The function is thus increasing on $[0, \pi/6) \cup (5\pi/6, 2\pi]$ and decreasing on $(\pi/6, 5\pi/6)$.

(d) The second derivative is $f''(x) = -\cos x$. This is never undefined, so the possible inflection points occur only when $f''(x) = 0$; that is, when $-\cos x = 0$; that is, when $x = \pi/2, 3\pi/2$.

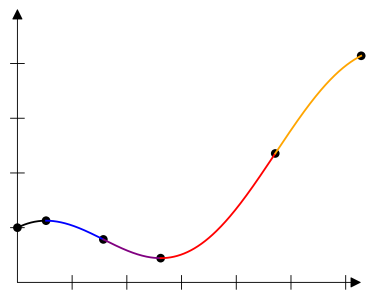
(e) To find the intervals of concavity, we choose values between each endpoint and/or critical point, and substitute into the second derivative:

$f''(x)$	-	-	0	+	0	-	-
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

The function is thus concave up on $(\pi/2, 3\pi/2)$ and decreasing on $[0, \pi/2) \cup (3\pi/2, 2\pi]$.

(f) There are no vertical asymptotes of f, because we never encounter division by zero. There are no horizontal asymptotes of f, because $\lim_{x\to\pm\infty} f$ does not converge to any one point.

(g) Different colors in the graph below highlight the information. (You would not need different colors on the graph.) We found y-values for all points of interest. The (first) black section of the graph runs from $x = 0$ to $x = \pi/6$ and is increasing and concave down. The (second) blue section of the graph runs from $x = \pi/6$ to $x = \pi/2$ and is decreasing and concave down. The (third) purple section of the graph runs from $x = \pi/2$ to $x = 5\pi/6$ and is decreasing and concave up. The (fourth) red section of the graph runs from $x = 5\pi/6$ to $x = 3\pi/2$ and is increasing and concave up. The (fifth) orange section of the graph runs from $x = 3\pi/2$ to $x = 2\pi$ and is increasing and concave down.



1.6. Compute a linear approximation of $\sin(0.75)$ by starting at the point $x = \pi/4$.

Solution: We are using $f(x) = \sin x$. The linear approximation of the curve uses the line tangent to the curve at the given point. The x-value we will use is given as $x_0 = \pi/4$, and the y-value is $y_0 = f(x_0) = f(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$. The slope of this line is the derivative. So, we need $f'(x) = \cos x$ and in particular we need $f'(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$. The equation of the line tangent to f at $x = \pi/2$ is thus

$$y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \cdot \left(x - \frac{\pi}{4}\right) \quad \Longrightarrow \quad y = \frac{\sqrt{2}}{2} \cdot \left(x + 1 - \frac{\pi}{4}\right).$$

So our approximation is

$$y = \frac{\sqrt{2}}{2} \cdot \left(0.75 + 1 - \frac{\pi}{4}\right) \approx 0.6821 .$$

In fact, a calculator will tell you that $\sin(0.75) \approx 0.6816$, so the approximation is rather good!

1.7. Use three steps of Newton's Method to approximate a solution to

$$x^5 + 2x = 1\,,$$

starting from the initial approximation $x_0 = 1$. Is your final answer accurate to three decimal places? Why or why not?

Solutions: Move every term in the equation to one side, obtaining

$$x^5 + 2x - 1 = 0\,.$$

Now we are solving for a root of $f(x) = x^5 + 2x - 1$. Its derivative is $f'(x) = 5x^4 + 2$. Newton's Method tells us to iterate with

$$x_{\text{next}} = x_{\text{curr}} - \frac{f(x_{\text{curr}})}{f'(x_{\text{curr}})}\,,$$

or

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}\,.$$

Hence we have

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= 1 - \frac{f(1)}{f'(1)} = 1 - \frac{2}{7} \approx 0.714286 \\
x_2 &= 0.714286 - \frac{f(0.714286)}{f'(0.714286)} = 0.714286 - \frac{0.6145068}{3.3015431} \approx 0.528159 \\
x_3 &= 0.528159 - \frac{f(0.528159)}{f'(0.528159)} = 0.528159 - \frac{0.0974163}{2.3890709} \approx 0.487383\,.
\end{aligned}$$

This answer is not even accurate to one decimal place, because none of the decimals are repeating!

(Were we to allow it to proceed one or two additional steps, it would in fact start repeating: x_4 repeats the first two digits of x_3, and x_5 repeats the first six digits of x_4.)