MAT 167 TEST 3 FORM A (APPLICATIONS OF DERIVATIVES)

Free-response questions

Directions: These questions must be answered in your blue exam book. You may use a graphing calculator to check your work, but I must see all necessary steps, as well as appropriate explanations when necessary. "Bald answers" receive at most one point.

1. *(10 pts)* Find y' if $\sin(x-y) = x\cos y$, and use it to find the equation of the line tangent to the curve at $x = 0$.

Solution: Differentiate with respect to x, remembering the chain rule and the product rule:

$$\cos(x-y)\cdot\underbrace{\left(1-\frac{dy}{dx}\right)}_{\text{chain}} = \underbrace{1\cdot\cos y + x\cdot(-\sin y)\ \underbrace{\cdot\frac{dy}{dx}}_{\text{chain}}}_{\text{product}}\ .$$

On the left, distribute the $\cos(x-y)$:

$$\cos(x-y) - \frac{dy}{dx}\cdot\cos(x-y) = \cos y - x\sin y\cdot\frac{dx}{dy}\ .$$

Bring everything with dy/dx to one side, everything without it to the the other:

$$x\sin y\cdot\frac{dy}{dx} - \frac{dy}{dx}\cdot\cos(x-y) = \cos y - \cos(x-y)\ .$$

Factor the common dy/dx:

$$\frac{dy}{dx}\left[x\sin y - \cos(x-y)\right] = \cos y - \cos(x-y)\ .$$

Divide to isolate dy/dx:

$$\frac{dy}{dx} = \frac{\cos y - \cos(x-y)}{x\sin y - \cos(x-y)}\ .$$

To find the slope, we need to substitute x into the derivative. Ordinarily, we'd need to substitute y, as well, but that's not actually necessary this time:

$$\frac{dy}{dx} = \frac{\cos y - \cos(0-y)}{0\sin y - \cos(0-y)} = \frac{\cos y - \cos(-y)}{0-\cos(-y)} = \frac{\cos y - \cos y}{-\cos y} = 0\ .$$

(We substituted $\cos(-y) = \cos y$ because cosine is an even function.)

Since the slope is 0, the line is horizontal. To find a y-value, we substitute $x = 0$ into the original equation:

$$\sin(0-y) = 0\cdot\cos y \quad\Longrightarrow\quad \sin(-y) = 0 \quad\Longrightarrow\quad y = -\arcsin 0 \quad\Longrightarrow\quad y = 0\ .$$

(There are other solutions, as well, but we only need one.) So the equation of the tangent line is

$$y - 0 = 0(x-0) \quad\Longrightarrow\quad y = 0\ .$$

2. *(10 pts)* Compute a linear approximation of $\sqrt[3]{2}$ from the point $x = 1$.

Solution: We want to approximate $f(2)$ when $f(x) = \sqrt[3]{x}$, using a tangent line placed at $x = 1$. We need a point and a slope. The point is $x = 1$, $y = f(1) = \sqrt[3]{1} = 1$. For the slope, we first need

$$f'(x) = \frac{1}{3}x^{-2/3}.$$

The slope is then $f'(1) = (1/3) \cdot 1^{-2/3} = 1/3$. Hence the line is

$$y - 1 = \frac{1}{3}(x-1) \quad \Longrightarrow \quad y = \frac{x}{3} + \frac{2}{3}.$$

An approximation of $f(2)$ is thus

$$y = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.$$

3. Let $f(x) = e^x - 3x$.

(a) *(15 pts)* Find the first three approximations of a root of f, using Newton's Method, starting from the point $x = 1$.

Solution: We need $f'(x) = e^x - 3$. Using the formula $x_{i+1} = x_i - f(x_i)/f'(x_i)$ and the starting point $x_0 = 1$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{e-3}{e-3} = 1 - 1 = 0$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{1-0}{1-3} = -\frac{1}{-2} = \frac{1}{2}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{1}{2} - \frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)} = \frac{1}{2} - \frac{0.14872127}{-1.35127873} \approx 0.61.$$

(b) *(5 pts)* How would you decide whether the approximation is correct to the hundredths place? Is that the case with your answer to part (a)?

Solution: We decide whether the approximation is correct to the hundredths place by checking whether the first two digits after the decimal have stabilized; that is, they no longer change in the iteration. In part (a), that has not happened yet; even the first digit after the decimal is changing.

4. Let $f(x) = e^x - 3x$, as in #3. We want to graph $f(x)$ on the interval $[-1, 2]$.

(a) *(5 pts)* List all the critical point(s) of f on the interval, if any.

Solution: We need $f'(x) = e^x - 3$. Set it to zero and solve for x:

$$e^x - 3 = 0 \quad \Longrightarrow \quad e^x = 3 \quad \Longrightarrow \quad x = \ln 3.$$

You can also list the endpoints. You could also give an approximation for $\ln 3$, such as 1.0986, as long as I saw how you obtained it. **Just writing 1.0986 is not enough for full credit**, since you can figure that out from looking at a picture on a graphing calculator..

(b) *(5 pts)* List all the inflection point(s) of f on the interval, if any.

Solution: We need $f''(x) = e^x$. Set it to zero and solve for x:

$$e^x = 0 \quad \Longrightarrow \quad \text{...uh, that never happens.}$$

So there are no inflection points.

(c) *(5 pts)* On what interval(s) is f increasing? On what interval(s) is f decreasing?
Solution: We need to check values of $f'(x)$ between critical points and each other, and between critical points and endpoints.

$f'(x)$	-	-	0	+	+
x	-2		$\ln 3$		1

We see that the curve is decreasing on $[-2, \ln 3)$ and increasing on $(\ln 3, 1]$.

(d) *(5 pts)* On what interval(s) is f concave up? On what interval(s) is f concave down?
Solution: We need to check values of $f''(x)$ between critical points... and that's it, because there are no possible inflection points.

$f''(x)$	+	+	+
x	-2		1

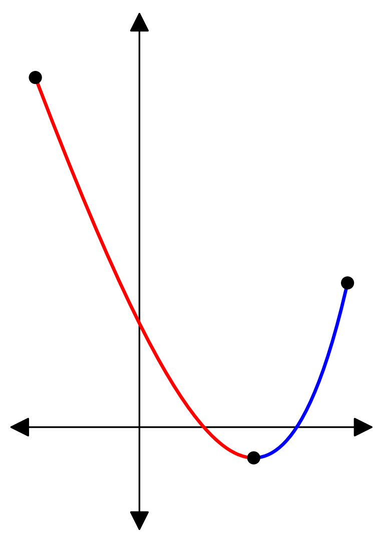
We see that the curve is concave up on $[-2, 1]$.

(e) *(5 pts)* Find the minima and maxima of f on $[-1, 2]$, and classify each as local or global.
Solution: We need to evaluate f at the endpoints and the critical points to find the largest and smallest y-values.

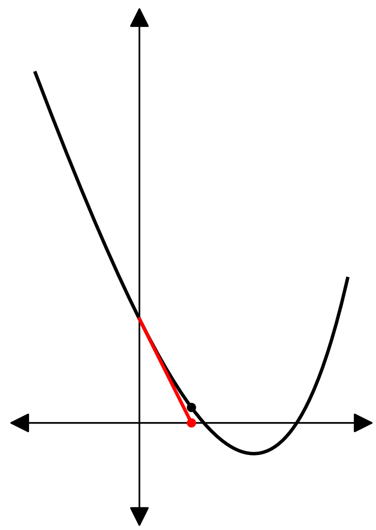
$$f(-2) \approx 6.135$$
$$f(\ln 3) \approx -0.296$$
$$f(1) \approx -0.282$$

The global maximum is at $x = -2$, the global minimum is at $x = \ln 3$, and a local maximum is at $x = 1$.

(f) *(5 pts)* Sketch a graph of f using the information you found in parts (a)–(e).
Solution: We need to combine the information from parts (a)–(e). We color red the segment of the curve that is decreasing but concave up, and we color blue the segment of the curve that is increasing but concave up.



4. *(Bonus)* Let $f(x) = e^x - 3x$, as in #3 and #4. Suppose I make a linear approximation of $f(0.5)$ from $x = 0$.
   (a) *(2 pts)* Is my approximation an overestimate or an underestimate?
   Solution: Most everyone who tried this attempted to perform a linear approximation, but that's too much work. *Just think about the graph!* If we perform a linear approximation of $f(0.5)$ from $x = 0$, we have the following graph:



   The graph makes it plain that the y-value of the linear approximation is lower than the actual value.
   (b) *(3 pts)* How does concavity explain this?
   Solution: The curve is decreasing and concave up on the interval $[0, 0.5]$, so it is bending upward, so the tangent line will lie below the curve.