MAT 167 TEST 2 FORM A (DERIVATIVES)

Directions: Solve each required problem **on a separate sheet of paper. Use pencil and show all work;** I deduct points for using pen or skipping important steps. **You must shut off your cell phone.** Some problems are worth more than others. Take your time; **quality is preferred to quantity.** I encourage you to ask questions.

1. Let $f(x) = 3x$.
   (a) Use the *intuitive* or *geometric* definition of the derivative to explain why $f'(x) = 3$ at every value of x.
   The derivative of f at a point is the rate of change of f at that point. The graph of f is a line with slope 3. "Slope" is the rate of change of a line, and is constant. Hence, the derivative is 3.
   (b) Use the *precise* definition of the derivative to show that $f'(x) = 3$ at every value of x.
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{3(x+h) - 3x}{h} \\ &= \lim_{h\to 0} \frac{\cancel{3x} + 3h - \cancel{3x}}{h} \\ &= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}} \\ &= \lim_{h\to 0} 3 \\ &= 3\,. \end{aligned}$$

2. Use *properties* of derivatives to compute the derivatives of the following functions.
   (a) $f(x) = xe^x$
   By the product shortcut, the $mx + b$ shortcut, and the e^x shortcut,
$$f'(x) = \left(\frac{d}{dx}x\right)e^x + x\left(\frac{d}{dx}e^x\right) = 1 \cdot e^x + x \cdot e^x = e^x + xe^x\,.$$
   (b) $g(x) = x^2 - {}^3/_x$
   By the sum shortcut and the x^n shortcut,
$$g'(x) = 2x + \frac{3}{x^2}\,.$$
   (We use the fact that we can rewrite $g(x) = x^2 - 3x^{-1}$.)

(c) $h(x) = \sin(2x) - 3\tan x$

By the sum shortcut, the constant multiple shortcut, two trigonometric shortcuts, and the Chain Rule,

$$h'(x) = \underbrace{\cos(2x)}_{\text{derivative of } \sin} \cdot \underbrace{2}_{\text{derivative of } 2x} - 3\sec^2 x = 2\cos(2x) - 3\sec^2 x\,.$$

(d) $G(x) = \dfrac{(2x+1)^3}{x^5 e^x}$

By the quotient shortcut, the product shortcut, the power shortcut, Chain Rule, and the $mx + b$ shortcut,

$$G'(x) = \frac{\left[\frac{d}{dx}\left((2x+1)^3\right) \cdot x^5 e^x\right] - \left[(2x+1)^3 \left(\frac{d}{dx}\left(x^5 e^x\right)\right)\right]}{\left(x^5 e^x\right)^2}$$

$$= \frac{\left[\left(3(2x+1)^2 \cdot 2\right) \cdot x^5 e^x\right] - \left[(2x+1)^3 \left(5x^4 e^x + x^5 e^x\right)\right]}{x^{10} e^{2x}} \qquad (1)$$

$$= \frac{6x^5 (2x+1)^2 e^x - (2x+1)^3 \left(5x^4 e^x + x^5 e^x\right)}{x^{10} e^{2x}}$$

$$= \frac{x^4 e^x \left[6x(2x+1)^2 - (2x+1)^3 (5+x)\right]}{x^{10} e^{2x}}$$

$$= \frac{6x(2x+1)^2 - (2x+1)^3 (5+x)}{x^6 e^x}\,.$$

(You can stop at equation (1) on the test.)

3. Use *properties* of derivatives to prove that $d/dx\,(\tan x) = \sec^2 x$.

By a trigonometric property and the quotient shortcut,

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\left(\frac{d}{dx}\sin x\right)\cos x - \sin x \left(\frac{d}{dx}\cos x\right)}{(\cos x)^2}$$

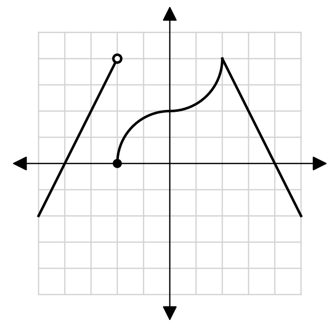
$$= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

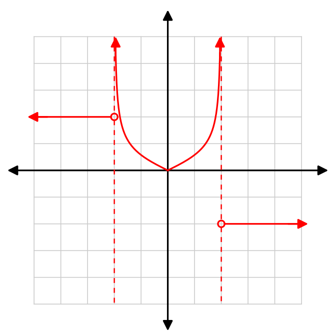
$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x\,.$$

4. Use the graph of $f(x)$ given at right to answer the following questions. (Each gray line marks a unit of 1.)



   (a) At what point(s) is f discontinuous?
   f is discontinuous wherever we see a hole, jump, or asymptote. There is one jump in this graph, at $x = -2$. There are no holes or asymptotes.
   (b) At what point(s) if f nondifferentiable?
   f is nondifferentiable whenever it is not smooth. This graph is not smooth at $x = -2$ (discontinuity) and at $x = 2$ (kink).
   (c) Sketch a graph of $f'(x)$.
   We find the graph of f' by estimating the slope of the tangent line at various points on the graph. The leftmost line segment has slope 2, so f' has a horizontal line $y = 2$ on that interval. The rightmost line segment likewise has slope -2, so f' has a horizontal line $y = -2$ on that interval. In between, we move from an apparently vertical tangent line at $x = -2$ to an apparently horizontal tangent line at $x = 0$, then back to an apparently vertical tangent line at $x = 2$.



5. Find the equation of the line tangent to $y = -2(3x+1)^2$ at $x = 0$.
   We can find any line using the point-slope equation, $y - y_0 = m(x - x_0)$. We are given that $x_0 = 0$. We can find y_0 by substitution:
$$y_0 = -2(3 \cdot 0 + 1)^2 = -2 \times 1 = -2 .$$
   The slope m is by definition the derivative of the given function at the given point. The derivative is
$$y' = \underbrace{-2}_{\text{const. mult.}} \cdot \underbrace{2(3x+1)^1}_{\text{power}} \cdot \underbrace{3}_{\text{chain}} ,$$
   and at $x_0 = 0$ we have
$$y' = -2 \cdot 2(3 \cdot 0 + 1) \cdot 3 = -12 .$$
   Hence the equation of the tangent line is
$$y - (-2) = -12(x - 0) ,$$
   or in point-slope form,
$$y = -12x - 2 .$$

6. **Insolation** describes the amount of sunlight an area of land receives at a given moment. It is measured in Langleys, Ly for short.[1] Suppose the insolation at a given location in Mississippi on a given day is $I = 17\sin(t \cdot \pi/14)$, where t is the number of hours after 6am.
   (a) Approximately how much insolation is the location receiving at 9am? Use correct units!
   The insolation is $I = 17\sin(3 \cdot \pi/14) \approx 10.59933\,\text{Ly}$.
   (Notice that $t = 3$ because t represents the number of hours after 6am, and at 9am we are 3 hours after 6am. Don't use $t = 9$; that would be wrong!)
   (b) Approximately how fast is the insolation changing at 9am? Use correct units!
   "How fast" asks for rate of change, so we need the derivative,
$$I' = \underbrace{17}_{\text{const. mult.}} \cdot \underbrace{\cos\left(t \cdot \frac{\pi}{14}\right)}_{\text{deriv. of sin}} \cdot \underbrace{\frac{\pi}{14}}_{\text{chain}} = \frac{17\pi}{14}\cos\left(\frac{\pi}{14}t\right).$$
   At $t = 3$,
$$I'(3) = \frac{17\pi}{14}\cos\left(\frac{\pi}{14} \cdot 3\right) \approx 2.9852\,\text{Ly}/\text{hr}.$$
   (Notice that the unit is Langleys per hour, because the derivative is dy/dx, and in this case "y" represents Langleys, while "x" represents hours after 9am.)
   (c) Is the insolation increasing or decreasing at 9am? Explain your answer.
   The insolation is increasing, because I' is positive.

7. **Bonus.** If $f(x) = \cos x$, find $f^{(275)}(x)$. Show your work for full credit.
   While I probably would ask a different bonus, I'm not telling the answer here. See me if you want to know.

[1] If you're curious, $1\text{Ly} = 41840\,\text{J}/\text{m}^2 \approx 11.622\,\text{Wh}/\text{m}^2$, but that's not useful to the problem.