MAT 167 TEST 1 FORM B (LIMITS)

Free-response questions

Directions: These questions must be answered in your blue exam book. You may use a graphing calculator to check your work, but I must see all necessary steps, as well as appropriate explanations when necessary. "Bald answers" receive at most one point.

1. *(5 pts each)*
   Give (a) a precise definition of the statement, "f is continuous at $x = a$," and
   You can find the limit of f at $x = a$ by substituting a for x. Or, written symbolically, $\lim_{x\to a} f(x) = f(a)$.

   (b) an intuitive or geometric definition of the statement, "$\lim_{x\to a} f(x) = L$."
   As the x-value approaches a from both sides, the y-value of f approaches L.

2. *(5 pts each)*
   Evaluate the following limits, if they exist. If they do not exist, state this. Justify your answers.

   (a) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$

$$\lim_{x\to 9} \frac{\sqrt{x}-3 \nearrow 0}{x-9 \searrow 0} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 9} \frac{x-9}{(x-9)\left(\sqrt{x}+3\right)} = \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}\,.$$

   (b) $\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{\left(x^2+2xh+h^2\right) - x^2}{h} = \lim_{h\to 0} \frac{(2x+h)\,h}{h} = \lim_{h\to 0} (2x+h) = 2x\,.$$

   (c) $\lim_{x\to\infty} \frac{4x^3+5}{2x^3-9}$

$$\lim_{x\to\infty} \frac{4x^3+5 \nearrow \infty}{2x^3-9 \searrow \infty} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{4+\frac{5}{x^3} \nearrow 4+0}{2-\frac{9}{x^3} \searrow 2-0} = \frac{4}{2} = 2\,.$$

3. *(10 pts each)*

(a) Use the Method of Bisection to approximate a root of the function $x^5 + 2x + 1$ on the interval $(-1, 0)$ in 3 steps.

We start with $a = -1$, $b = 0$, $n = 3$, $f(x) = x^5 + 2x + 1$.

Initialize $c = a = -1$, $d = b = 0$, $i = n = 3$.

Let $e = (c + d)/2 = -1/2$. By substitution, $f(e) = -1/32 < 0$, so we replace $c = e = -1/2$ and decrement i to 2.

Let $e = (c + d)/2 = -1/4$. By substitution, $f(e) \approx 0.5 > 0$, so we replace $d = e = -1/4$ and decrement i to 1.

Let $e = (c + d)/2 = -3/8$. By substitution, $f(e) \approx 0.24 > 0$, so we replace $d = e = -3/8$ and decrement i to 0.

At this point, $i = 0$, so the algorithm returns $[c, d] = [-1/2, -3/8]$.

(b) Explain how we know that we can use the Method of Bisection to approximate a root of the given function.

The function f is a polynomial and thus continuous. If we check $f(a) = -2$ and $f(b) = 1$, we see that they have opposite signs. Since 0 lies between the y-values, the Intermediate Value Theorem tells us that a root of f lies between a and b.

4. *(Bonus, 10 pts)*
Problem 4 asked you to use at most 3 steps. How many steps would it take to guarantee that the answer was accurate to 5 places?

The Method of Bisection guarantees that if you give it an interval $[a, b]$, then the result $[c, d]$ satisfies $|c - d| = |a - b|/2^n$, where n is the number of steps. We want a result accurate to 5 decimal places, so we need $|c - d| < 10^{-5}$. In problem 4, $a = 0$ and $b = 1$, so by substitution,

$$\frac{|a-b|}{2^n} < 10^{-5} \quad \Longrightarrow \quad |1-0| \times 10^5 < 2^n \quad \Longrightarrow \quad \log_2 10^5 < n \quad \Longrightarrow \quad 17 \le n\,.$$