MAT 167 TEST 1 FORM A (LIMITS)

FREE-RESPONSE QUESTIONS

Directions: These questions must be answered in your blue exam book. You may use a graphing calculator to check your work, but I must see all necessary steps, as well as appropriate explanations when necessary. "Bald answers" receive at most one point.

1. *(5 pts each)*
   Give (a) an intuitive or geometric definition of the statement, "f is continuous at $x = a$," and
   The graph of f has no hole, jump, or asymptote at $x = a$.
   (b) a precise definition of the statement, "f is continuous at $x = a$."
   You can find the limit of f at $x = a$ by substituting a for x. Or, written symbolically, $\lim_{x\to a} f(x) = f(a)$.

2. *(10 pts each)*
   Evaluate the following limits, if they exist. If they do not exist, state this. Justify your answers.
   (a) $\displaystyle\lim_{x\to 2} \frac{x^2-4}{x-2}$

$$\lim_{x\to 9} \frac{x^2-4}{x-2}{}^{\nearrow 0}_{\searrow 0} = \lim_{x\to 9} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 9} x+2 = 4\,.$$

   (b) $\displaystyle\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

$$\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}{}^{\nearrow 0}_{\searrow 0} &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0} \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}\,.
\end{aligned}$$

   (c) $\displaystyle\lim_{x\to\infty} \frac{2x^3+5}{4x^3-9}$

$$\lim_{x\to\infty} \frac{2x^3+5}{4x^3-9}{}^{\nearrow\infty}_{\searrow\infty} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{2+\frac{5}{x^3}}{4-\frac{9}{x^3}}{}^{\nearrow 2+0}_{\searrow 4-0} = \frac{2}{4} = \frac{1}{2}\,.$$

3. *(10 pts each)*

(a) Use the Method of Bisection to approximate a root of the function $x^5 + 2x - 1$ on the interval $(0, 1)$ in 3 steps.

We start with $a = 0$, $b = 1$, $n = 3$, $f(x) = x^5 + 2x - 1$.

Initialize $c = a = 0$, $d = b = 1$, $i = n = 3$.

Let $e = (c + d)/2 = 1/2$. By substitution, $f(e) = 1/32 > 0$, so we replace $d = e = 1/2$ and decrement i to 2.

Let $e = (c + d)/2 = 1/4$. By substitution, $f(e) \approx -0.5 > 0$, so we replace $c = e = 1/4$ and decrement i to 1.

Let $e = (c + d)/2 = 3/8$. By substitution, $f(e) \approx -0.24 > 0$, so we replace $c = e = 3/8$ and decrement i to 0.

At this point, $i = 0$, so the algorithm returns $[c, d] = [3/8, 1/2]$.

(b) Explain how we know that we can use the Method of Bisection to approximate a root of the given function.

The function f is a polynomial and thus continuous. If we check $f(a) = 1$ and $f(b) = 2$, we see that they have opposite signs. Since 0 lies between the y-values, the Intermediate Value Theorem tells us that a root of f lies between a and b.

4. *(Bonus, 10 pts)*

Problem 4 asked you to use at most 3 steps. How many steps would it take to guarantee that the answer was accurate to 5 places?

The Method of Bisection guarantees that if you give it an interval $[a, b]$, then the result $[c, d]$ satisfies $|c - d| = |a - b|/2^n$, where n is the number of steps. We want a result accurate to 5 decimal places, so we need $|c - d| < 10^{-5}$. In problem 4, $a = 0$ and $b = 1$, so by substitution,

$$\frac{|a-b|}{2^n} < 10^{-5} \quad \Longrightarrow \quad |1-0| \times 10^5 < 2^n \quad \Longrightarrow \quad \log_2 10^5 < n \quad \Longrightarrow \quad 17 \le n\,.$$