More examples of Related Rates and L'Hôpital's Rule

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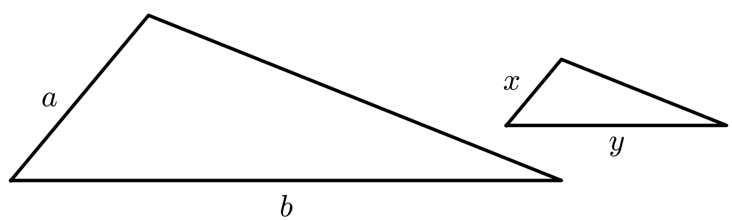
Related Rates

1. If two triangles are similar, then their sides are proportional, so that in the diagram below, we have

$$\frac{a}{b} = \frac{x}{y}\,. \tag{1}$$

Suppose that when $a = 50\,\mathrm{m}$ and $b = 72\,\mathrm{m}$, we know that $x = 12\,\mathrm{m}$, a is increasing by $10\,{}^{\mathrm{m}}/_{\mathrm{s}}$, b is decreasing by $12\,{}^{\mathrm{m}}/_{\mathrm{s}}$, and x is decreasing by $18\,{}^{\mathrm{m}}/_{\mathrm{s}}$. If the triangles remain similar at all times, how is y changing?



Solution: None of the quantities remains constant, so we cannot substitute them yet. We take the derivative with respect to time, which requires the quotient rule:

$$\frac{\frac{da}{dt}\cdot b - a\cdot\frac{db}{dt}}{b^2} = \frac{\frac{dx}{dt}\cdot y - x\cdot\frac{dy}{dt}}{y^2}\,.$$

Now let's go ahead and substitute the values we know:

$$\frac{10\cdot 72 - 50\cdot(-12)}{72^2} = \frac{-18\cdot y - 12\cdot\frac{dy}{dt}}{y^2}\,. \tag{2}$$

(We use -12 and -18 because b and x are decreasing.) It looks as if we are at an impasse, because we have two unknowns: y and ${}^{dy}/_{dt}$. However, from the proportion (1) we know that

$$\frac{50}{72} = \frac{12}{y} \quad\Longrightarrow\quad 50y = 12\times 72 \quad\Longrightarrow\quad y = \frac{432}{25} = 17.28\,\mathrm{m}\,.$$

Substitute this into (2), and we can solve for y:

$$\frac{10 \cdot 72 - 50 \cdot (-12)}{72^2} = \frac{-18 \cdot 17.28 - 12 \cdot \frac{dy}{dt}}{17.28^2} \quad \Longrightarrow \quad \frac{dy}{dt} = -32.256 \; .$$

Since the value is negative, y is decreasing at a rate of 32.256 m/s.

2. A pipe leaking from under ground feeds a circular puddle in the road. If the area is changing at a constant rate of 5 $\mathrm{cm}^2/\mathrm{min}$, at what rate is the circle's radius increasing when it is 12 cm?

   Solution: First we related the quantities, $A = \pi r^2$. Neither A nor r is constant, so we cannot substitute their values yet. We take the derivative with respect to time, which gives us

$$\frac{dA}{dt} = \pi \cdot 2r \underbrace{\cdot \frac{dr}{dt}}_{\text{chain}} \; .$$

   Now we substitute the values we know:

$$5 = \pi \cdot 2 \cdot 12 \cdot \frac{dr}{dt} \; .$$

   We solve for dr/dt to find that

$$\frac{dr}{dt} = \frac{5}{24\pi} \approx 0.07 \; .$$

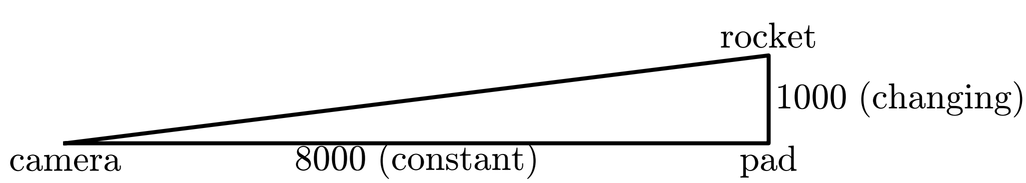
   The radius is increasing at 0.07 cm/min.

3. A television camera is positioned 8000 ft from the base of a rocket launching pad. The camera's angle of elevation must change at the correct rate to keep the rocket in sight. Also, the camera's focusing mechanism must take into account the increasing distance from the camera to the rising rocket. Assume the rocket rises vertically, and its velocity is 900 ft/s when its height is 1000 ft.

   (a) How fast is the distance from the camera to the rocket changing at that moment?

   (b) If the camera remains aimed at the rocket, how fast is the camera's angle of elevation changing at that moment?

   Solution: The diagram below illustrates the situation. We have a right triangle because the rocket rises vertically.



(a) We need merely worry about the distances. Let a be the distance from the camera to the pad, b the distance from the pad to the rocket, and c the distance from the camera to the rocket. We can relate these values using the equation

$$a^2 + b^2 = c^2 .$$

While $a = 8000$ is constant, $b = 1000$ is not, so we can substitute only for a. We have

$$8000^2 + b^2 = c^2 . \tag{3}$$

Take the derivative with respect to time, which gives us

$$0 + 2b \cdot \frac{db}{dt} = 2c \cdot \frac{dc}{dt} .$$

Substitute for the values we know:

$$2 \cdot 1000 \cdot 900 = 2c \cdot \frac{dc}{dt} . \tag{4}$$

We want to know dc/dt, but without knowing c we cannot proceed. Fortunately, we know from equation (3) that

$$a^2 + b^2 = c^2 \quad \Longrightarrow \quad 8000^2 + 1000^2 = c^2 \quad \Longrightarrow \quad c \approx 8062 .$$

Substituting that into (4),

$$2 \cdot 1000 \cdot 900 = 2 \cdot 8062 \cdot \frac{dc}{dt} \quad \Longrightarrow \quad \frac{dc}{dt} \approx 111.6 .$$

So the distance between the camera and the rocket increases at a rate of roughly 111.6 ft/s.

(b) The angle of elevation is the angle at the camera; call it γ ("gamma"). We can relate this quantity to the others using trigonometry:

$$\tan \gamma = \frac{b}{a} .$$

While $a = 8000$ is constant, $b = 1000$ is not, so we can substitute only for a. We have

$$\tan \gamma = \frac{b}{8000} . \tag{5}$$

Take the derivative with respect to time, which gives us

$$\sec^2 \gamma \cdot \frac{d\gamma}{dt} = \frac{1}{8000} \cdot \frac{db}{dt} .$$

Substitute for the values we know:

$$\sec^2 \gamma \cdot \frac{d\gamma}{dt} = \frac{1}{8000} \cdot 900 . \tag{6}$$

That's not especially helpful; we need to know γ. Fortunately, we know from equation (5) that

$$\tan \gamma = \frac{1000}{8000} \quad \Longrightarrow \quad \gamma = \tan^{-1} \frac{1}{8} \approx 0.12345 .$$

Substituting that into (6), we have

$$\sec^2 \left(\tan \frac{1}{8} \right) \cdot \frac{d\gamma}{dt} = \frac{9}{80} \quad \Longrightarrow \quad \frac{65}{64} \cdot \frac{d\gamma}{dt} = \frac{9}{80} \quad \Longrightarrow \frac{d\gamma}{dt} = 36/326 \approx 0.11 .$$

The angle of elevation must increase at a rate of approximately 0.11 radians/second.

L'Hôpital's Rule

Evaluate the following limits. (You need to worry about only 1–4 on the final, but you may need 5–7 later in life.)

1. $\displaystyle \lim_{x \to 3^+} \frac{17 \ln (x-3)}{28\,(x-3)}$

   Solution:

   $$\lim_{x \to 3^+} \frac{17 \ln (x-3)^{\nearrow -\infty}}{28\,(x-3)_{\searrow 0}} = -\infty\,.$$

   Don't think you have to use L'Hôpital's Rule just because that's a recent technique used! You should use only what applies, and L'Hôpital's Rule *does not apply* because the form is neither $^0/_0$ nor $^{\pm\infty}/_{\pm\infty}$.

2. $\displaystyle \lim_{x \to 1} \frac{4 \ln x}{5 \tan (\pi x)}$

   Solution:

   $$\lim_{x \to 1} \frac{4 \ln x}{5 \tan (\pi x)} \overset{\text{L'H}}{=} \lim_{x \to 1} \frac{4 \cdot \frac{1}{x}^{\nearrow 4}}{5 \sec^2 (\pi x) \cdot \pi_{\searrow 5\pi}} = \frac{4}{5\pi}\,.$$

3. $\displaystyle \lim_{t \to 5} \frac{t^2 - 25}{4t^2 + 4t - 120}$

   Solution:

   $$\lim_{t \to 5} \frac{t^2 - 25^{\nearrow 0}}{4t^2 + 4t - 120_{\searrow 0}} \overset{\text{L'H}}{=} \lim_{t \to 5} \frac{2t^{\nearrow 10}}{8t + 4_{\searrow 44}} = \frac{10}{44} = \frac{5}{22}\,.$$

4. $\displaystyle \lim_{\alpha \to 0} \frac{\sin (18\alpha)}{\sin (12\alpha)}$

   Solution:

   $$\lim_{\alpha \to 0} \frac{\sin (18\alpha)^{\nearrow 0}}{\sin (12\alpha)_{\searrow 0}} \overset{\text{L'H}}{=} \lim_{\alpha \to 0} \frac{\cos (18\alpha) \cdot 18^{\nearrow 18}}{\cos (12\alpha) \cdot 12_{\searrow 12}} = \frac{18}{12} = \frac{3}{2}\,.$$

5. $\displaystyle \lim_{\theta \to \frac{\pi}{2}^+} (\sec \theta - \tan \theta)$

   Solution:

   $$\lim_{\theta \to \frac{\pi}{2}^-} (\sec \theta - \tan \theta) \to \infty - \infty\,.$$

   This is an indeterminate form. We need to rewrite it as one of the two forms that work with L'Hôpital's Rule. We can do this by resorting to trigonometric identities and fractions:

   $$\lim_{\theta \to \frac{\pi}{2}^-} (\sec \theta - \tan \theta) = \lim_{\theta \to \frac{\pi}{2}^-} \left(\frac{1}{\cos \theta} - \frac{\sin \theta}{\cos \theta} \right) = \lim_{\theta \to \frac{\pi}{2}^-} \left(\frac{1 - \sin \theta}{\cos \theta} \right)^{\nearrow 0}_{\searrow 1} = 0\,.$$

We didn't even need L'Hôpital's Rule! It sufficed simply to rewrite the original expression as fractions.

(Don't conclude too quickly that $\infty - \infty = 0$, as the next problem shows.)

6. $\displaystyle \lim_{x\to 1^+} \left(\frac{12}{\ln x} - \frac{12}{x-1} \right)$

Solution:

$$\lim_{x\to 1} \left(\frac{12}{\ln x} - \frac{12}{x-1} \right) \to \infty - \infty \, .$$

This is an indeterminate form. We need to rewrite it as one of the two forms that work with L'Hôpital's Rule. We can do this by resorting to fractions:

$$\lim_{x\to 1} \left(\frac{12}{\ln x} - \frac{12}{x-1} \right) = \lim_{x\to 1} \left(\frac{12\,(x-1) - 12 \ln x}{(x-1)\ln x} \right)^{\nearrow 0}_{\searrow 0} \overset{\text{L'H}}{=} \lim_{x\to 1} \frac{12 \cdot 1 - 12 \cdot \frac{1}{x}}{1 \cdot \ln x + (x-1) \cdot \frac{1}{x}}^{\nearrow 0}_{\searrow 0} \, .$$

We seem to have ended up with $0/0$ again. Let's simplify a little bit — we have fractions within fractions, and that's never pleasant.

$$\lim_{x\to 1} \left(\frac{12}{\ln x} - \frac{12}{x-1} \right) = \lim_{x\to 1} \frac{12x - 12}{x \ln x + (x-1)}^{\nearrow 0}_{\searrow 0} \, .$$

We still have $0/0$. No cause to panic yet; apply L'Hôpital's Rule a second time.

$$\lim_{x\to 1} \left(\frac{12}{\ln x} - \frac{12}{x-1} \right) \overset{\text{L'H}}{=} \lim_{x\to 1} \frac{12}{1 \cdot \ln x + x \cdot \frac{1}{x} + 1}^{\nearrow 12}_{\searrow 2} = \frac{12}{2} = 6 \, .$$

7. $\displaystyle \lim_{x\to\infty} \left(1 + \frac{3}{x} \right)^{2x}$

Solution: (As an aside, this problem is pretty fundamental to finance and economics.)

$$\lim_{x\to\infty} \left(1 + \frac{3}{x} \right)^{2x} \to 1^{\infty} \, .$$

This is an indeterminate form. We need to rewrite it as one of the two forms that work with L'Hôpital's Rule. Since we have a variable in the exponent, we resort to logarithms. Let

$$y = \left(1 + \frac{3}{x} \right)^{2x} \, .$$

Then

$$\ln y = \ln \left(1 + \frac{3}{x} \right)^{2x} = 2x \cdot \ln \left(1 + \frac{3}{x} \right) \, .$$

Let's consider

$$\lim_{x\to\infty} (\ln y) = \lim_{x\to\infty} \left[2x \cdot \ln \left(1 + \frac{3}{x} \right) \right] \to \infty \cdot \ln(1+0) = \infty \cdot 0 \, .$$

This is also an indeterminate form.[1] We need to rewrite it as one of the two forms that work with L'Hôpital's Rule. We'll resort to fractions, moving $2x$ to the denominator via a reciprocal:

$$\lim_{x\to\infty} (\ln y) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)^{\nearrow 0}}{\frac{1}{2x}_{\searrow 0}} \stackrel{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{3}{x}} \cdot -\frac{3}{x^2}}{-\frac{1}{2x^2}} = \lim_{x\to\infty} \left[6\left(\frac{1}{1+\frac{3}{x}}\right)\right] \to 6 \cdot \frac{1}{1+0} = 6\,.$$

We found a limit! However, it isn't the limit we originally wanted. We've found the limit of $\ln y$; we want the limit of y. Fortunately, it's easy to convert from $\ln y$ to y:

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e^6\,.$$

[1] I can't remember if I mentioned it in class. If $0 \cdot \infty$ is not listed, you should include it in the list of indeterminate forms that I gave.