# QUIZ 3 SOLUTIONS

MAT 167

p. 86 #9. **Analyzing infinite limits graphically.** The graph of $f$ in the figure [not shown here] has vertical asymptotes at $x = 1$ and $x = 2$. Analyze the following limits:

(a) $\lim_{x\to 1^-} f(x) = \infty$ (b) $\lim_{x\to 1^+} f(x) = \infty$ (c) $\lim_{x\to 1} f(x) = \infty$

(d) $\lim_{x\to 2^-} f(x) = \infty$ (e) $\lim_{x\to 2^+} f(x) = -\infty$ (f) $\lim_{x\to 2} f(x)$ does not exist

p. 86 #41. **Explain why or why not.** Determine whether the following statements are true or false and give an explanation or counterexample.

(a) The line $x = 1$ is a vertical asymptote of the function $f(x) = {}^{x^2-7x+6}/_{x^2-1}$.
**False.** The limit of the denominator approaches 0, and the limit of the numerator does, as well. When we encounter $^0/_0$, there is more work to be done; factor the numerator as $(x-1)(x+6)$ and the limit becomes

$$\lim_{x\to 1} \frac{(x-1)(x+6)}{x-1} = \lim_{x\to 1}(x+6) = 7.$$

The limit exists *and is finite*, so it is not an asymptote.

(b) The line $x = -1$ is a vertical asymptote of the function $f(x) = {}^{x^2-7x+6}/_{x^2-1}$.
**True.** The limit of the denominator approaches 0, while the limit of the numerator approaches a nonzero value: 14, to be precise. When we encounter $^{\text{nonzero}}/_0$, the limit from the left or the right is some sort of infinity. We don't really care which one here; it suffices to show there is a vertical asymptote at $x = 1$.

(c) If $g$ has a vertical asymptote at $x = 1$ and $\lim_{x\to 1^+} g(x) = \infty$, then $\lim_{x\to 1^-} g(x) = \infty$.
**False.** The left-hand limit could be $-\infty$, or could even be a finite number. Knowing the right-hand limit is infinite is all we need for an asymptote. To be absolutely clear, the function $g(x) = {}^1/_{x-1}$ would serve as an example: the right-sided limit is $\infty$ while the left-sided limit is $-\infty$.

p. 86 #53. **Limits with a parameter.** Let $f(x) = \dfrac{x^2 - 7x + 12}{x - a}$.

(a) For what values of $a$, if any, does $\lim_{x\to a^+} f(x)$ equal a finite number?
The denominator approaches 0, so the only way to have a finite limit is if the numerator also approaches 0. The numerator factors as $(x-4)(x-3)$, so it approaches 0 only if $x = 4$ or $x = 3$. The only possible values of $x = a$ that have a finite limit are thus $x = 3$ and $x = 4$; if we actually try them out, we

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see that

$$\lim_{x\to 3^+}\frac{x^2-7x+12}{x-3}=\lim_{x\to 3^+}\frac{(x-3)(x-4)}{x-3}=\lim_{x\to 3^+}(x-4)=-1$$

and

$$\lim_{x\to 4^+}\frac{x^2-7x+12}{x-4}=\lim_{x\to 4^+}\frac{(x-3)(x-4)}{x-4}=\lim_{x\to 4^+}(x-3)=1,$$

confirming our reasoning.

(b) For what values of $a$, if any, does $\lim_{x\to a^+} f(x)$ equal $\infty$?
The denominator approaches 0, so to have an infinite limit of some sort, we merely need the numerator to approach a nonzero value: for that, any $a\neq 3,4$ will do. However, we want *positive* infinity, so we have to be a little more careful. When $x$ approaches $a$ from the right, the $x$-values are larger than $a$. So the denominator will always be positive. To have a limit of *positive* infinity, the numerator's values must also be positive.

- This is true when $x>4$, as the factorization $(x-3)(x-4)$ becomes $(+)\cdot(+)$.
- This is also true when $x<3$, as the factorization becomes $(-)\cdot(-)$.
- This is false when $3<x<4$, as the factorization becomes $(+)\cdot(-)$.

So the values of $a$ for which the limit is $\infty$ are those in the interval $(-\infty,3)\cup(4,\infty)$.

(c) For what values of $a$, if any, does $\lim_{x\to a^+} f(x)$ equal $-\infty$?
We've basically answered this in part (b); the values for $a$ for which the limit is $-\infty$ are those in the interval $(3,4)$.

p. 98 #9. Evaluate the limit.

$$\lim_{x\to\infty}\left(3+\frac{10}{x^2}\right)\to 3+10\cdot 0=3.$$

p. 98 #13. Evaluate the limit.
We need to squeeze this function. We know that

$$-1\leq\cos(\text{anything})\leq 1.$$

If that's true for anything, it's certainly true for $x^5$, so

$$-1\leq\cos x^5\leq 1.$$

We care about ${}^{\cos x^5}/_{\sqrt{x}}$, so let's multiply all three sides by ${}^1/_{\sqrt{x}}$ to obtain the desired function in the middle:

$$-\frac{1}{\sqrt{x}}\leq\frac{\cos x^5}{\sqrt{x}}\leq\frac{1}{\sqrt{x}}\,.$$

Recall that $\sqrt{x}=x^{1/2}$. We pointed out in class that ${}^1/_{x^p}\to 0$ as $x\to\infty$ so long as $p>0$. In this problem, $p={}^1/_2$, so certainly ${}^1/_2>0$. So the left- and right-hand sides of the inequality approach 0. By the Squeeze Theorem, the "middle side" must also approach 0. Hence

$$\lim_{x\to\infty} \frac{\cos x^5}{\sqrt{x}} \to 0.$$

p. 98 #27. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Then give the horizontal asymptote of $f$ (if any).

$$\lim_{x\to\infty} \frac{6x^2-9x+8}{3x^2+2} \begin{matrix}\to\\ \to\end{matrix} \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{6x^2-9x+8}{3x^2+2} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{6-9/x+8/x^2}{3+2/x^2} \begin{matrix}\to\\ \to\end{matrix} \frac{6-0+0}{3+0} = 2.$$

The other limit is the same. There is a horizontal asymptote at $y=2$.

p. 98 #29. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Then give the horizontal asymptote of $f$ (if any).

$$\lim_{x\to\infty} \frac{3x^3-7}{x^4+5x^2} \begin{matrix}\to\\ \to\end{matrix} \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{3x^3-7}{x^4+5x^2} \cdot \frac{1/x^4}{1/x^4} = \lim_{x\to\infty} \frac{3/x-7/x^4}{1+5/x^2} \begin{matrix}\to\\ \to\end{matrix} \frac{0-0}{1+0} = 0.$$

The other limit is the same. There is a horizontal asymptote at $y=0$.

p. 98 #53. (a) Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, and then identify any horizontal asymptotes.

$$\begin{aligned}\lim_{x\to\infty} \frac{2x^3+10x^2+12x}{x^3+2x^2} & \begin{matrix}\to\\ \to\end{matrix} \frac{\infty}{\infty} \\ &= \lim_{x\to\infty} \frac{2x^3+10x^2+12x}{x^3+2x^2} \cdot \frac{1/x^3}{1/x^3} \\ &= \lim_{x\to\infty} \frac{2+10/x+12/x^2}{1+2/x} \\ & \begin{matrix}\to\\ \to\end{matrix} \frac{2+0+0}{1+0} \\ &= 2.\end{aligned}$$

The other limit is the same. There is a horizontal asymptote at $y=2$.

(b) Find the vertical asymptotes. For each vertical asymptote $x=a$, evaluate $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.

We first look for division by zero. Set the denominator to zero:

$$x^3+2x^2=0 \quad\Rightarrow\quad x^2(x+2)=0 \quad\Rightarrow\quad x=0,-2.$$

For a vertical asymptote, the numerator must approach a nonzero value, possibly after reduction. When $x = -2$, we have

$$\lim_{x\to -2^+} \frac{2x^3 + 10x^2 + 12x}{x^3 + 2x^2} \begin{array}{c}\to\\ \to\end{array} \frac{0}{0}$$

$$= \lim_{x\to -2^+} \frac{2x(x+2)(x+3)}{x^2(x+2)}$$

$$= \lim_{x\to -2^+} \frac{2(x+3)}{x}$$

$$\begin{array}{c}\to\\ \to\end{array} \frac{-4}{4} = -1 \neq \pm\infty.$$

The left-hand limit is the same. When $x = 0$, we have

$$\lim_{x\to 0^+} \frac{2x^3 + 10x^2 + 12x}{x^3 + 2x^2} \begin{array}{c}\to\\ \to\end{array} \frac{0}{0} \quad = \lim_{x\to 0^+} \frac{2(x+3)}{x} \begin{array}{c}\to\\ \to\end{array} \frac{6}{0^+} \to \infty$$

and

$$\lim_{x\to 0^-} \frac{2x^3 + 10x^2 + 12x}{x^3 + 2x^2} \begin{array}{c}\to\\ \to\end{array} \frac{0}{0} \quad = \lim_{x\to 0^-} \frac{2(x+3)}{x} \begin{array}{c}\to\\ \to\end{array} \frac{6}{0^-} \to -\infty.$$

So there is only one vertical asymptote, and it occurs at $x = 0$.

p. 98 #71. **Steady states.** If a function $f$ represents a system that varies in time, the existence of $\lim_{x\to\infty} f(x)$ means that the system reaches a steady state (or equilibrium). For the population of a culture of tumor cells given by $p(t) = {}^{3500}/_{t+1}$, determine if a steady-state exists and give the steady-state value.

To decide the question, we must evaluate the limit:

$$\lim_{x\to\infty} \frac{3500}{t+1} \begin{array}{c}\to\\ \to\end{array} \frac{3500}{\infty} \quad = 0.$$

So yes, there is a steady state at population 0, when all the cells die.

p. 98 #77. **Looking ahead to sequences.** A sequence is an infinite, ordered list of numbers that is often defined by a function. ...The limit of such a sequence is $\lim_{n\to\infty} f(n)$, provided the limit exists. Find the limit of the sequence $\{0, {}^1/_2, {}^2/_3, {}^3/_4, \ldots\}$, which is defined by $f(n) = {}^{n-1}/_n$, or state that the limit does not exist.

Evaluating the limit, we find that

$$\lim_{n\to\infty} \frac{n-1}{n} \begin{array}{c}\to\\ \to\end{array} \frac{\infty}{\infty} \quad = \lim_{n\to\infty} \frac{n-1}{n} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1 - \frac{1}{n}}{1} \begin{array}{c}\to\\ \to\end{array} \frac{1}{1} = 1.$$

The limit exists, and is 1.