QUIZ 2 SOLUTIONS

MAT 167

p. 109 #53(a). **Intermediate Value Theorem and related rates.** Suppose \$5000 is invested in a savings account for 10 years (120 monhts), with an annual interest rate of r, compounded monthly. The amount of money in the account after 10 years is $A(r) = 5000(1 + r/12)^{120}$. Use the Intermediate Value Theorem to show there is a value of r in $(0, 0.08)$ — an interest rate between 0% and 8% — that allows you to reach your savings goal of \$7000 in 10 years.

$A(0) = 5000(1 + 0/12)^{120} = 5000$ and $A(0.08) = 5000(1 + 0.08/12)^{120} = 11058.20$. Since $A(0) < 7000 < A(0.08)$, A is a polynomial, and polynomials are continuous everywhere, there must be a value of r in $(0, 0.08)$ that gives $A(r) = 7000$.

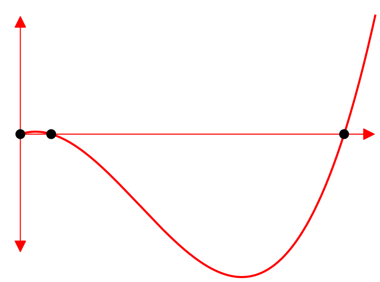
p. 109 #57. (a) Use the Intermediate Value Theorem to show that the equation $x^3 - 5x^2 + 2x = -1$ has a solution on the interval $(-1, 5)$.

Let $f(x)$ represent the left-hand side of the equation. Notice that $f(-1) = (-1)^3 - 5(-1)^2 + 2(-1) = -8$ while $f(5) = 5^3 - 5(5^2) + 2(5) = 10$. Since $f(-1) < -1 < f(5)$, f is polynomial, and polynomials are continuous everywhere, there must be a value of x in $(-1, 5)$ that gives $f(x) = -1$.

(b) Use a graphing utility to find all the solutions to the equation in that interval.

There are in fact three roots: one at 0, one near 0.43485, and one near 4.56155.

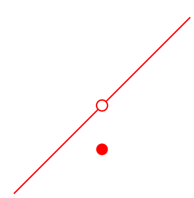
(c) Illustrate your answers with an appropriate graph.



p. 109 #79. (a) Sketch the graph of a function that is not continuous at 1, but is defined at 1.

(b) Sketch the graph of a function that is not continuous at 1, but has a limit at 1.

There is more than one way to answer this question, but one graph can illustrate both:



Date: Fall 2016.

p. 109 #97. Classify the discontinuities in $h(x) = {}^{x^3-4x^2+4x}/_{x(x-1)}$ at the points $x=0$ and $x=1$. See exercises 91–92.

Consider the limits at each point:

$$\lim_{x\to 1} \frac{x^3-4x^2+4x}{x(x-1)} \begin{matrix} \to \\ \to \end{matrix} \frac{\text{nonzero}}{0}$$
$$\lim_{x\to 0} \frac{x^3-4x^2+4x}{x(x-1)} \begin{matrix} \to \\ \to \end{matrix} \frac{0}{0}$$

The point $x=1$ has an **asymptotic discontinuity**: if we examine the limits from the left and right, we will obtain some sort of infinity. The point $x=0$ needs further investigation, so we factor, simplify, and try again:

$$\lim_{x\to 0} \frac{x(x-2)^2}{x(x-1)} = \lim_{x\to 0} \frac{(x-2)^2}{x-1} \begin{matrix} \to \\ \to \end{matrix} \frac{4}{-1} = -4.$$

As described in the introduction to exercises 91–92, this is a **removable discontinuity**.

p. 77 #27. Evaluate the limit.

$$\lim_{x\to 1} \frac{5x^2+6x+1}{8x-4} \begin{matrix} \to \\ \to \end{matrix} \frac{12}{4} = 3.$$

p. 77 #33. **One-sided limits.** Let

$$f(x) = \begin{cases} x^2+1 & \text{if } x < -1 \\ \sqrt{x+1} & \text{if } x \geq -1. \end{cases}$$

Compute the following limits or state that they do not exist.

(a) $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \left(x^2+1\right) = 2.$

(b) $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \sqrt{x+1} = 0.$

(c) $\lim_{x\to -1} f(x)$ does not exist.

p. 77 #41. Evaluate the limit.

$$\lim_{x\to 4} \frac{x^2-16}{4-x} \begin{matrix} \to \\ \to \end{matrix} \frac{0}{0} \quad = \lim_{x\to 4} \frac{(x-4)(x+4)}{4-x} = \lim_{x\to 4} \frac{(x-4)(x+4)}{-(x-4)} = \lim_{x\to 4} \frac{x+4}{-1} \begin{matrix} \to \\ \to \end{matrix} \frac{8}{-1} = -8.$$

p. 77 #47. Evaluate the limit.

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \begin{matrix} \to \\ \to \end{matrix} \frac{0}{0} \quad = \lim_{x\to 9} \left(\frac{\sqrt{x}-3}{x-9} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} \right) = \lim_{x\to 9} \frac{x-9}{(x-9)\left(\sqrt{x}+3\right)} = \lim_{x\to 9} \frac{1}{\sqrt{x}+3} \begin{matrix} \to \\ \to \end{matrix} \frac{1}{6}.$$

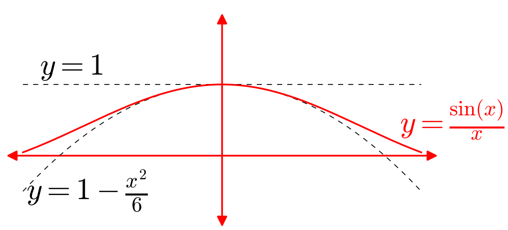
p. 77 #51. Evaluate the limit.

$$\lim_{h\to 0} \frac{\sqrt{16+h}-4}{h} \to \frac{0}{0}$$

$$\begin{aligned}\lim_{h\to 0} \frac{\sqrt{16+h}-4}{h} &= \lim_{h\to 0}\left(\frac{\sqrt{16+h}-4}{h}\cdot\frac{\sqrt{16+h}+4}{\sqrt{16+h}+4}\right)\\ &= \lim_{h\to 0}\frac{(16+h)-16}{h\left(\sqrt{16+h}+4\right)}\\ &= \lim_{h\to 0}\frac{1}{\sqrt{16+h}+4}\\ &\to \frac{1}{8}.\end{aligned}$$

p. 77 #57. **A sine limit by the Squeeze Theorem.** It can be shown that $1-x^2/6 \le \sin x/x \le 1$, for x near 0.

(a) Illustrate these inequalities with a graph.



(b) Use these inequalities to find $\lim_{x\to 0} \sin x/x$.

We are told that

$$1-\frac{x^2}{6} \le \frac{\sin x}{x} \le 1.$$

The left-hand function is a polynomial, hence continuous, and we can evaluate the limit by substitution:

$$\lim_{x\to 0}\left(1-\frac{x^2}{6}\right) = 1-\frac{0^2}{6} = 1.$$

The right-hand function is a constant, so the limit is itself: 1. The limits at $x=0$ of the left- and right-hand functions are identical. By the Squeeze Theorem,

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

p. 77 #67. Evaluate the limit, assuming k is constant.

$$\lim_{w \to -k} \frac{w^2 + 5kw + 4k^2}{w^2 + kw} \begin{array}{c}\to \\ \to\end{array} \frac{0}{0}$$

$$\begin{aligned}\lim_{w \to -k} \frac{w^2 + 5kw + 4k^2}{w^2 + kw} &= \lim_{w \to -k} \frac{(w + 4k)(w + k)}{w(w + k)} \\ &= \lim_{w \to -k} \frac{w + 4k}{w} \\ &\begin{array}{c}\to \\ \to\end{array} \frac{3k}{-k} = -3.\end{aligned}$$

p. 77 #71. Calculate the limit using the factorization formula

$$x^n - a^n = (x - a)\left(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}\right),$$

where n is a positive integer and a is a real number.

$$\lim_{x \to 1} \frac{x^6 - 1}{x - 1} \begin{array}{c}\to \\ \to\end{array} \frac{0}{0}$$

$$\begin{aligned}\lim_{x \to 1} \frac{x^6 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x - 1)\left(x^5 + x^4 + x^3 + x^2 + x + 1\right)}{x - 1} \\ &= \lim_{x \to 1} \left(x^5 + x^4 + x^3 + x^2 + x + 1\right) \\ &= 6.\end{aligned}$$