REVIEW 2 SOLUTIONS, PART 2

MAT 167

p. 167 #29. **Derivatives of other trigonometric functions** Verify the derivative formula $\frac{d}{dx}(\cot x) = -\csc^2 x$ using the Quotient Rule.

Recall from trigonometry that $\cot x = {}^{\cos x}/_{\sin x}$. This has the form of a quotient, so

$$\begin{aligned}\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) &= \frac{\left(\frac{d}{dx}\cos x\right)\cdot \sin x - \cos x \cdot \left(\frac{d}{dx}\sin x\right)}{\sin^2 x} \\ &= \frac{(-\sin x)\cdot \sin x - \cos x \cdot (\cos x)}{\sin^2 x} \\ &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\,.\end{aligned}$$

Now recall from trigonometry the "Pythagorean identity," $\sin^2 x + \cos^2 x = 1$. We can rewrite the above as

$$\frac{d}{dx}(\cot x) = -\frac{1}{\sin^2 x} = -\csc^2 x.$$

p. 167 #63. **Equations of tangent lines**

(a) Find the equation of the line tangent to the curve $y = 1 + 2\sin x$ at $x = {}^{\pi}/_{6}$.

The equation of a line is $y - y_0 = m(x - x_0)$. We know $x_0 = {}^{\pi}/_{6}$, so $y_0 = 1 + 2\sin({}^{\pi}/_{6}) = 1 + 2 \cdot {}^{1}/_{2} = 2$. The slope of the tangent line is the derivative, so we need to compute

$$y' = 0 + 2 \cdot \cos x\,;$$

at $x = {}^{\pi}/_{6}$ we have $y' = 2 \cdot {}^{\sqrt{3}}/_{2} = \sqrt{3}$. Hence the tangent line is

$$y - 2 = \sqrt{3}\left(x - \frac{\pi}{6}\right).$$

(The book's answer is correct, but an abomination to avoid writing.)

(b) Use a graphing utility to plot the curve and the tangent line.

(omitted)

p. 167 #69. **Velocity of an oscillator** An object oscillates along a vertical line…

(a) Graph the position function, for $0 \le t \le 10$.

(omitted)

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(b) Find the velocity of the oscillator, $v(t) = y'(t)$.
Since $y(t) = 30(\sin t - 1)$, we must have $v(t) = y'(t) = 30(\cos t - 0) = 30\cos t$.

(c) Graph the velocity function, for $0 \le t \le 10$.
(omitted)

(d) At what times and positions is the velocity zero?
The velocity is zero when $0 = 30\cos t$, or $0 = \cos t$, or $t = \pi/2 + \pi k$ for any $k \in \mathbb{Z}$ (that is, "for any integer k"). The positions at these times are

$$30\left(\sin\frac{\pi}{2} - 1\right) = 0,$$
$$30\left(\sin\frac{3\pi}{2} - 1\right) = -60,$$
$$30\left(\sin\frac{5\pi}{2} - 1\right) = 0,$$
$$30\left(\sin\frac{7\pi}{2} - 1\right) = -60,$$
$$\vdots$$

and so forth. (All positions are in centimeters; all times are in second.)

(e) At what times and positions is the velocity a maximum?
If we look at the graph of the velocity function, we see that velocity is a maximum when $t = \pi k$ for any $k \in \mathbb{Z}$. The positions at these times are

$$30(\sin 0 - 1) = -30,$$
$$30(\sin \pi - 1) = -30,$$
$$30(\sin 2\pi - 1) = -30,$$
$$30(\sin 3\pi - 1) = -30,$$
$$\vdots$$

and so forth. (All positions are in centimeters; all times are in seconds.)

(f) The acceleration of the oscillator is $a(t) = v'(t)$. Find and graph the acceleration function.
Since $v(t) = 30\cos t$, we must have $a(t) = v'(t) = 30(-\sin t) = -30\sin t$. *(Graph omitted.)*

p. 187 #13. **Version 1 of the Chain Rule** Use Version 1 of the Chain Rule to calculate dy/dx for $y = \sqrt{x^2+1}$.

Version 1 of the Chain Rule is in Leibniz notation,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

In our problem, $y = \sqrt{u} = u^{1/2}$ where $u = x^2 + 1$. So

$$\frac{dy}{dx} = \underbrace{\frac{1}{2}u^{-\frac{1}{2}}}_{\frac{dy}{du}} \cdot \underbrace{2x}_{\frac{du}{dx}} = u^{-\frac{1}{2}} \cdot x = \underbrace{(x^2+1)^{-\frac{1}{2}} \cdot x}_{\text{I am happy here}} = \frac{x}{\sqrt{x^2+1}}.$$

As far as I'm concerned, you may also use Version 2 of the Chain Rule to solve this.

p. 187 #35. **Similar-looking composite functions** Two composite functions are given that look similar, but in fact are quite different. Identify the inner function $u = g(x)$ and the outer function $y = f(u)$; then evaluate ${}^{dy}/_{dx}$ using the Chain Rule.

(a) $y = \cos^3 x$

This actually means $y = (\cos x)^3$, so $u = \cos x$ and $y = u^3$. The Chain Rule tells us that

$$\frac{dy}{dx} = \underbrace{3u^2}_{\frac{dy}{du}} \cdot \underbrace{(-\sin x)}_{\frac{du}{dx}} = \underbrace{-3(\cos x)^2 \sin x}_{\text{I am happy here}} = -3 \sin x \cos^2 x.$$

(b) $y = \cos x^3$

This actually means $y = \cos\left(x^3\right)$, so $u = x^3$ and $y = \cos u$. The Chain Rule tells us that

$$\frac{dy}{dx} = \underbrace{-\sin u}_{\frac{dy}{du}} \cdot \underbrace{3x^2}_{\frac{du}{dx}} = -3x^2 \sin\left(x^3\right).$$

p. 187 #37. **Chain Rule using a table** Let $h(x) = f(g(x))$ and $p(x) = g(f(x))$. Use the table *(omitted)* to compute the following derivatives.

(a) $h'(3) = f'(g(3)) \cdot g'(3) = f'(1) \cdot 20 = 5 \cdot 20 = 100$

(b) $h'(2) = f'(g(2)) \cdot g'(2) = f'(5) \cdot 10 = -10 \cdot 10 = -100$

(c) $p'(4) = g'(f(4)) \cdot f'(4) = g'(1) \cdot (-8) = 2 \cdot (-8) = -16$

(d) $p'(2) = g'(f(2)) \cdot f'(2) = g'(3) \cdot 2 = 20 \cdot 2 = 40$

(e) $h'(5) = f'(g(5)) \cdot g'(5) = f'(2) \cdot 20 = 2 \cdot 20 = 40$

p. 187 #55. **Combining Rules** Use the Chain Rule combined with other differentiation rules to find the derivative of $y = \left(\frac{x}{x+1}\right)^5$.

The inside is $u = {}^{x}/_{x+1}$, and the outside is $y = u^5$, so

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \underbrace{5u^4}_{\text{power}} \cdot \underbrace{\frac{1 \cdot (x+1) - x \cdot (1+0)}{(x+1)^2}}_{\text{quotient}} = \underbrace{5\left(\frac{x}{x+1}\right)^4 \cdot \frac{1}{(x+1)^2}}_{\text{I am happy here}} = \frac{5x^4}{(x+1)^6}.$$

p. 187 #82. **Vibrations of a spring** Suppose an object of mass m...

(a) Find ${}^{dy}/_{dt}$, the velocity of the mass. Assume k and m are constant.

For the function $y = y_0 \cos\left(t\sqrt{k/m}\right)$, the inside is $u = t\sqrt{k/m}$, and the outside is $y = y_0 \cos u$, so

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = y_0(-\sin u) \cdot \left(1 \cdot \sqrt{\frac{k}{m}}\right) = -\left(y_0\sqrt{\frac{k}{m}}\right) \sin\left(t\sqrt{\frac{k}{m}}\right).$$

(b) How would the velocity be affected if the experiment were repeated with four times the mass on the end of the spring?

We need to replace m by $4m$. In this case, the oscillation is halved in frequency and in distance, because

$$\sqrt{\frac{k}{4m}} = \frac{1}{2} \cdot \sqrt{\frac{k}{m}}.$$

(c) How would the velocity be affected if the experiment were repeated with a spring having four times the stiffness (k is increased by a factor of 4)?
We need to replace k by $4k$. In this case, the oscillation is doubled in frequency and in distance, because

$$\sqrt{\frac{4k}{m}} = 2\sqrt{\frac{k}{m}}.$$

(d) Assume that y has units of meters, t has units of seconds, m has units of kg, and k has units of kg/s^2. Show that the units of the velocity in part (a) are consistent.
The units for velocity should be m/s. If we look at what the derivative gave us, we have

$$y_0\sqrt{\frac{k}{m}} \quad \rightarrow \quad \mathrm{m} \cdot \sqrt{\frac{\mathrm{kg}/\mathrm{s}^2}{\mathrm{kg}}} = \mathrm{m} \cdot \sqrt{\frac{\cancel{\mathrm{kg}}/\mathrm{s}^2}{\cancel{\mathrm{kg}}}} = \mathrm{m} \cdot \sqrt{\frac{1}{\mathrm{s}^2}} = \mathrm{m} \cdot \frac{1}{\mathrm{s}} = \mathrm{m}/\mathrm{s}.$$

p. 187 #83. **Vibrations of a spring** Suppose an object of mass m...

(a) Find the second derivative d^2y/dt^2.
This is the derivative of the first derivative, which we found in part (a) of #82. For the function

$$v = -\left(y_0\sqrt{\frac{k}{m}}\right)\sin\left(t\sqrt{\frac{k}{m}}\right),$$

the inside is $u = t\sqrt{k/m}$, and the outside is $v = -y_0\sqrt{k/m}\sin u$, so

$$v' = \frac{dv}{dt} = \frac{dv}{du} \cdot \frac{du}{dt} = -y_0\sqrt{\frac{k}{m}} \cdot \cos(u) \cdot \left(1 \cdot \sqrt{\frac{k}{m}}\right) = -\frac{y_0 k}{m}\cos\left(t\sqrt{\frac{k}{m}}\right).$$

(b) Verify that $d^2y/dt^2 = -(k/m)y$.
We saw in part (a) that

$$\frac{d^2y}{dt^2} = v' = -\frac{y_0 k}{m}\cos\left(t\sqrt{\frac{k}{m}}\right).$$

We are given in the problem that $y = y_0 \cos\left(t\sqrt{k/m}\right)$. Examining how this appears in the line above, we see indeed that

$$\frac{d^2y}{dt^2} = -\frac{k}{m} \cdot k\cos\left(t\sqrt{\frac{k}{m}}\right) = -\frac{k}{m} \cdot y.$$