REVIEW 2 SOLUTIONS

MAT 167

p. 138 #15. **Equations of tangent lines by definition (1)**

(a) Use definition (1) (p. 128) to find the slope of the line tangent to the graph of $f(x) = x^2 - 5$ at $P = (3,4)$.

The slope is the derivative:

$$
\begin{aligned}
m = f'(3) &= \lim_{x\to 3} \frac{f(x) - f(3)}{x-3} \\
&= \lim_{x\to 3} \frac{\left(x^2-5\right)-4}{x-3} \\
&= \lim_{x\to 3} \frac{x^2-9}{x-3} \\
&= \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3} \\
&= \lim_{x\to 3} (x+3) \\
&= 6.
\end{aligned}
$$

(b) Determine an equation of the tangent line at P.

Using the point-slope form of the line,

$$y - y_0 = m(x - x_0) \quad \Longrightarrow \quad y - 4 = 6(x-3).$$

(c) Plot the graph of f and the tangent line at P.

(omitted)

p. 138 #25. **Equations of tangent lines by definition (2)**

(a) Use definition (2) (p. 129) to find the slope of the line tangent to the graph of $f(x) = \sqrt{x+3}$ at $P = (1,2)$.

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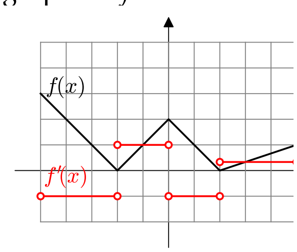
The slope is the derivative:

$$
\begin{aligned}
m = f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(1+h)+3}-\sqrt{1+3}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \times \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} \\
&= \lim_{h\to 0} \frac{(4+h)-4}{h\left(\sqrt{4+h}+2\right)} \\
&= \lim_{h\to 0} \frac{h}{h\left(\sqrt{4+h}+2\right)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} \\
&= \frac{1}{4}.
\end{aligned}
$$

(b) Determine an equation of the tangent line at P.
Using the point-slope form of the line, we get $y-2 = {}^1\!/_4(x-1)$.

p. 138 #47. Use the graph of f to sketch a graph of f'.



p. 138 #53. **Where is the graph continuous? Differentiable?** Use the graph of f in the figure (omitted) to do the following.

(a) Find the values of x in $(0,3)$ at which f is not continuous.
$x=1$ *(jump), and that's it.*

(b) Find the values of x in $(0,3)$ at which f is not differentiable.
$x=1$ *(discontinuous) and* $x=2$ *(corner).*

(c) Sketch a graph of f'.
(omitted)

p. 138 #65. **Power and energy** Energy is the capacity to do work… *(remainder omitted)*

(a) Estimate the power at $t = 10$ and $t = 20$ hr. Be sure to include units in your calculation.

Power is the derivative of energy, or, the slope of the line tangent to the energy curve. Using the graph of the energy curve provided in the book, we see that at $t = 10$ the power is roughly $\frac{325-0}{15-5} = \frac{325}{10} = 32.5$ kilowatt-hours per hour (or just plain "kilowatts") while at $t = 20$ the power is roughly $\frac{315-340}{25-20} = \frac{25}{5} = 5$ kilowatt-hours per hour (or just plain "kilowatts").

Notice how I made it explicit that I am finding these numbers by computing the slope between two points. You must do this for full credit. Answers that differ from mine, but are found reasonably, will be accepted at full credit.

(b) At what times on the interval $[0, 25]$ is the power zero?

Power is zero when the tangent line to energy is horizontal; that is, when $t = 6$, $t = 18$ or thereabouts.

(c) At what times on the interval $[0, 25]$ is the power a maximum?

Power is at a maximum when the tangent line to energy is positive and steepest; that is, when $t = 12$ or thereabouts.

p. 138 #79. **Determining the unknown constant** Let

$$f(x) = \begin{cases} 2x^2 & \text{if } x \leq 1 \\ ax - 2 & \text{if } x > 1. \end{cases}$$

Determine a value of a (if possible) for which $f'(1)$ exists.

For $f'(1)$ to exist, the left- and right-hand derivatives must exist and equal. That means we need

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1^+} \frac{f(x) - f(1)}{x-1}.$$

From the left, $f(x) = 2x^2$ and $f(1) = 2$, so the limit becomes

$$\lim_{x\to 1^-} \frac{2x^2 - 2}{x-1} = \lim_{x\to 1^-} \frac{2(x-1)}{x-1} = \lim_{x\to 1^-} 2 = 2.$$

From the right, $f(x) = ax - 2$ and $f(1) = a - 2$, so the limit becomes

$$\lim_{x\to 1^+} \frac{(ax-2)-(a-2)}{x-1} = \lim_{x\to 1^+} \frac{a(x-1)}{x-1} = \lim_{x\to 1^+} a = a.$$

For the left- and right-hand limits to be equal, we would need $a = 2$.

p. 148 #39. **Finding slope locations** Let $f(x) = x^2 - 6x + 5$.

(a) Find the values of x for which the slope of the curve $y = f(x)$ is 0.

The slope of the curve is the derivative, and $f'(x) = 2x - 6$. We want to know when this is 0. That occurs when $2x - 6 = 0$, or $x = 3$.

(b) Find the values of x for which the slope of the curve $y = f(x)$ is 2.

The slope of the curve is the derivative, and $f'(x) = 2x - 6$. We want to know when this is 2. That occurs when $2x - 6 = 2$, or $x = 4$.

p. 148 #45. **Higher-order derivatives** Find $f'(x)$, $f''(x)$, and $f^{(3)}(x)$ for $f(x) = 5x^4 + 10x^3 + 3x + 6$.

$f'(x) = 20x^3 + 30x^2 + 3$, $f''(x) = 60x^2 + 60x$, $f^{(3)}(x) = 120x + 60$

p. 148 #49. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

(a) The derivative $\frac{d}{dx}(10^5)$ equals $5 \cdot 10^4$.
False. The derivative of a constant is 0.

(b) The slope of a line tangent to $f(x) = e^x$ is never 0.
True. The graph of e^x is constantly decreasing, so its derivative is always negative.

(c) $\frac{d}{dx}(e^3) = e^3$
False. The derivative of a constant is 0.

(d) $\frac{d}{dx}(e^x) = xe^{x-1}$
False. The derivative of e^x is e^x. The power shortcut applies only when the base is variable and the power is constant.

(e) The nth derivative $\frac{d^n}{dx^n}(5x^3 + 2x + 5)$ equals 0, for any integer $n \geq 3$.
False. The third derivative is 30. Were $n > 3$ the statement would be true.

p. 148 #57. **Derivatives from a table** Use the table *(omitted)* to find $\frac{d}{dx}[1.5f(x)]\Big|_{x=2}$.

The derivative of $1.5f(x)$ is $1.5f'(x)$. From the table, we see that $f'(2) = 5$. Hence the derivative of $1.5f(x)$ at $x = 2$ is $1.5 \times 5 = 7.5$.

p. 148 #69. **Height estimate** The distance an object falls...

(a) Compute $d'(t)$. What units are associated with the derivative, and what does it measure?
Since $d(t)$ is given as $d(t) = 16t^2$, we know that $d'(t) = 32t$. Its units are in feet per second (determined by $\frac{\Delta d}{\Delta t}$, as d is measured in feet and t in seconds). It measure the rate of change of distance, or, the velocity.

(b) If it takes 6 s for a stone to fall to the ground, how high is the ledge? How fast is the stone moving when it strikes the ground (in mi/hr)?
If it takes 6 s for a stone to fall to the ground, then the ledge's height is $d(6) = 16 \cdot 6^2 = 576$ feet. The stone is moving at $d'(6) = 32 \cdot 6 = 192$ ft/s.

p. 158 #33. **Equations of tangent lines** $y = \frac{x+5}{x-1}$, $a = 3$

(a) Find an equation of the line tangent to the given curve at a.
We use the point-slope equation of the line. For $x_0 = a = 3$, we have $y_0 = \frac{3+5}{3-1} = 4$. For the slope, we need the derivative. Using the quotient rule, we find that

$$y' = \frac{(1+0)(x-1) - (x+5)(1-0)}{(x-1)^2} = -\frac{6}{(x-1)^2}.$$

By substituting x_0, we have $m = y'(3) = -6/4 = -3/2$. So the equation of the line tangent to the given curve at a is $y - 4 = -3/2(x-3)$.

(b) Use a graphing utility to graph the curve and the tangent line on the same set of axes.
(omitted)

p. 158 #51. **Population growth** Consider the population function

$$p(t) = \frac{800}{1 + 7e^{-0.2t}}.$$

(a) Find the instantaneous growth rate of the population, for $t \geq 0$.

We want the derivative. Using the quotient rule,

$$p'(t) = \frac{0 \cdot \left(1+7e^{-0.2t}\right) - 800\left(0+7\cdot(-0.2)\,e^{-0.2t}\right)}{(1+7e^{-0.2t})^2} = \frac{1120e^{-0.2t}}{(1+7e^{-0.2t})^2}.$$

(b) What is the instantaneous growth rate at $t = 5$?
We want

$$p'(5) = \frac{1120e^{-0.2\times 5}}{(1+7e^{-0.2\times 5})^2} = \frac{1120e^{-1}}{(1+7e^{-1})^2} \approx 32.2.$$

(c) Estimate the time when the instantaneous growth rate is the greatest.
If we look at a graph of $p'(t)$ *(omitted)* we see that the growth rate is greatest around $t = 10$.

(d) Evaluate and interpret $\lim_{t\to\infty} p'(t)$.
Since $e^{-0.2t} \to 0$ as $t \to \infty$, we have

$$\lim_{t\to\infty} \frac{1120e^{-0.2t}}{(1+7e^{-0.2t})^2} \quad \frac{\to 0}{\to 1} = 0.$$

Interpretation: in the long term, this population stops growing.

(e) Use a graphing utility to graph the population and its growth rate for $0 \le t \le 200$.
(omitted)

p. 158 #59. **Combining rules** Compute the derivative of $h(x) = \dfrac{xe^x}{x+1}$.

This requires a quotient rule, as well as a product rule when differentiating the numerator:

$$h'(x) = \frac{[1\cdot e^x + xe^x]\cdot(x+1) - xe^x\cdot(1+0)}{(x+1)^2} = \frac{\left(xe^x + e^x + x^2e^x + xe^x\right) - xe^x}{(x+1)^2} = \frac{e^x\left(x^2+x+1\right)}{(x+1)^2}.$$

p. 158 #81. **Electrostatic force** The magnitude of the electrostatic force… *(remainder omitted)*

(a) Find the instantaneous rate of change of the force with respect to the distance between the charges.
The instantaneous rate of change of the force is the derivative, so we want

$$F'(x) = kQq\cdot\left(-2x^{-3}\right) = -\frac{2kQq}{x^3}.$$

Notice how we used the facts that (a) k, Q, and q are constants, so we can treat them as constant multiples of $1/x^2$, and (b) $1/x^2 = x^{-2}$, so we can apply the power rule to compute the derivative.

(b) For two identical charges with $Q = q = 1\text{C}$, what is the instantaneous rate of change of the force at a separation of $x = 0.001\,\text{m}$?
By substitution (the problem tells us that $k = 9\times 10^9 \text{N}\cdot\text{m}^2/\text{C}^2$),

$$F'(.001) = -\frac{2\cdot 9\times 10^9\cdot 1\cdot 1}{(.001)^3} = \frac{18\times 10^9}{(10^{-3})^3} = 18\times 10^{18}.$$

(c) Does the magnitude of the instantaneous rate of change of the force increase or decrease with the separation? Explain.

"Magnitude" is a fancy word for "size," or "absolute value." To know how the absolute value of the instantaneous rate of change is changing, observe that

$$\left|F'(x)\right| = \frac{2kQq}{x^3}\ .$$

As x increases and the other values remain constant, $|F'(x)|$ will decrease, because we're dividing a constant by larger and larger values.