MAT 167 TEST 2 SOLUTIONS

1. From Form A

3. For each function or equation, compute the derivative.

(a) $y = 10\cos x - 2\arctan x$ (b) $G(y) = \left(\frac{y^2}{y+1}\right)^4$

(c) $g(u) = \sqrt{11\left(u^2+1\right)}$ (d) $p(x) = x\ln x$

(e) $x^2 + y^2 = \left(2x^2 + 2y^2 - x\right)^2$

Solutions:

(a) $y' = -10\sin x - \frac{2}{1+x^2}$

(b) Observe that G is a composition of functions; in particular, $G(u) = u^4$ where $u = \frac{y^2}{y+1}$. By the Chain Rule, $G'(y) = G'(u) \cdot u'$.

- To compute $G'(u)$, use the Power Rule and $G'(u) = 4u^3$.
- To compute u' , use the Quotient Rule and

$$u' = \frac{2y \cdot (y+1) - y^2 \cdot 1}{(y+1)^2} = \frac{y^2+2y}{(y+1)^2}.$$

- Thus

$$G'(y) = \left[4\left(\frac{y^2}{y+1}\right)^3\right] \cdot \frac{y^2+2y}{(y+1)^2}.$$

Remark. We do *not* need implicit differentiation here. G is a function of y, so the derivative is with respect to y, not with respect to x. Thus $\frac{dy}{dx}$ does not need to appear in the derivative.

(c) Observe that g is a composition of functions; in particular, $g(v) = \sqrt{11v}$ where $v = u^2+1$. (There are other ways to describe this composition.) By the Chain Rule, $g'(u) = g'(v) \cdot v'$.

- To compute $g'(v)$, notice that $g(v)$ contains a constant multiple: $g(v) = \sqrt{11} \cdot \sqrt{v}$. So use the Power Rule *and* the Constant Multiple Rule and $g'(v) = \sqrt{11} \cdot \frac{1}{2} v^{-\frac{1}{2}} = \frac{\sqrt{11}}{2\sqrt{v}}$.
- To compute v', use the Sum Rule, the Power Rule, and the Constant Rule and $v' = 2u$.
- Thus

$$g'(u) = \frac{\sqrt{11}}{2\sqrt{u^2+1}} \cdot 2u.$$

Remark. Notice that if u is already used and you need to identify a composition, you can always use another variable.

(d) To compute $p'(x)$, use the Product Rule and $p'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$.

(e) To compute y', we need implicit differentiation. (Here y depends on x.) So

$$\begin{aligned}
\frac{d}{dx}\left[x^2+y^2 = \left(2x^2+2y^2-x\right)^2\right] & \\
& \text{Right-hand side: Chain Rule!} \\
& \text{Let } u = 2x^2+2y^2-x; \\
& \text{then } u' = 4x+4y\cdot y'-1. \\
2x+2y\cdot y' &= 2\left(2x^2+2y^2-x\right)\cdot\left(4x+4y\cdot y'-1\right) \\
2x+2y\cdot y' &= 16x^3+16x^2y\cdot y'-4x^2 \\
&\quad +16xy^2+16y^3\cdot y'-4y^2 \\
&\quad -8x^2-8xy\cdot y'+2x \\
2x+2y\cdot y' &= \left(16x^3+16xy^2-12x^2-4y^2+2x\right) \\
&\quad +\left(16x^2y+16y^3-8xy\right)\cdot y' \\
\left[2y-\left(16x^2y+16y^3-8xy\right)\right]\cdot y' &= \left(16x^3+16xy^2-12x^2-4y^2+\cancel{2x}\right)-\cancel{2x} \\
y' &= \frac{16x^3+16xy^2-12x^2-4y^2}{2y-\left(16x^2y+16y^3-8xy\right)}.
\end{aligned}$$

4. Suppose the equation of a shock absorber is modeled by the equation $s(t) = 2e^{-0.9}\sin(2\pi t)$, where s is measured in centimeters and t in seconds.
   (a) Find the velocity, v, after t seconds.
   (b) What are the units of the velocity?
   (c) Is the shock absorber compressing or expanding at time $t = 3$?

Solutions:

(a) $v(t) = s'(t) = 2e^{-0.9}\cos(2\pi t)\cdot 2\pi$.
Notice that $e^{-0.9}$ is a *constant* (t does not appear in the exponent!) so we use the Constant Multiple Rule, along with the Chain Rule with $u = 2\pi t$ because $\cos(2\pi t)$ is a composition of functions.

(b) The units of velocity are cm/sec (change in s per change in t).

(c) At $t = 3$, $v(t) = 2e^{-0.9}\cos(2\pi\cdot 3)\cdot 2\pi = 2e^{-0.9}\cos(6\pi) = 2e^{-0.9}\cdot 1 > 0$ so the absorber is expanding.

5. Differentiate

$$y = \frac{\left(x^{12}-2x^8+3\right)^7}{\left(x^2-1\right)^5\left(x^3+x+2\right)^3}.$$

Hint: Use logarithmic differentiation.

Solution:

Using properties of logarithms, we have

$$
\begin{aligned}
\ln y &= \ln\left[\frac{\left(x^{12}-2x^8+3\right)^7}{\left(x^2-1\right)^5\left(x^3+x+2\right)^3}\right] \\
&= \ln\left(x^{12}-2x^8+3\right)^7 - \ln\left(\left(x^2-1\right)^5\left(x^3+x+2\right)^3\right) \\
&= \ln\left(x^{12}-2x^8+3\right)^7 - \left[\ln\left(x^2-1\right)^5 + \ln\left(x^3+x+2\right)^3\right] \\
&= 7\ln\left(x^{12}-2x^8+3\right) - \left[5\ln\left(x^2-1\right) + 3\ln\left(x^3+x+2\right)\right].
\end{aligned}
$$

Now differentiate both sides with respect to x, and we have (dont' forget the Chain Rule!)

$$
\begin{aligned}
\frac{1}{y}\cdot y' &= 7\cdot\frac{1}{x^{12}-2x^8+3}\cdot\left(12x^{11}-16x^7\right) \\
&\quad - \left[5\cdot\frac{1}{x^2-1}\cdot 2x + 3\cdot\frac{1}{x^3+x+2}\cdot\left(3x^2+1\right)\right] \\
y' &= \frac{\left(x^{12}-2x^8+3\right)^7}{\left(x^2-1\right)^5\left(x^3+x+2\right)^3}\cdot\left[7\cdot\frac{1}{x^{12}-2x^8+3}\cdot\left(12x^{11}-16x^7\right)\right. \\
&\quad \left. - \left[5\cdot\frac{1}{x^2-1}\cdot 2x + 3\cdot\frac{1}{x^3+x+2}\cdot\left(3x^2+1\right)\right]\right].
\end{aligned}
$$

2. From Form B

2. A mass on a spring vibrates according to the equation $x(t) = 4\cos(3\pi(t+2))$, where t is measured in seconds and x is measured in centimeters.
   (a) Find the velocity, v, and the acceleration, a, at time t.
   (b) What are the units of the velocity and the acceleration?
   (c) In what direction is the mass moving at time $t = 12$?
   (d) At what time(s) is the mass stationary? (That is, when is its velocity zero?)

Solutions:

(a) Use the Constant Multiple Rule and the Chain Rule ($u = 3\pi\cdot(t+2)$) to compute

$$v(t) = x'(t) = 4\cdot[-\sin(3\pi(t+2))\cdot 3\pi]$$

and then

$$a(t) = v'(t) = -12\pi\cos(3\pi(t+2))\cdot 3\pi.$$

(b) The units of velocity are cm/sec. The units of acceleration are cm^2/sec.
(c) At $t = 12$,

$$v(12) = -12\pi\sin(3\pi(12+2)) = 4\cdot 0 = 0,$$

so the mass is *not* moving at time $t = 12$.

(d) The mass is stationary when $v(t) = 0$. Thus we need to solve

$$\begin{aligned} -12\pi \sin(3\pi(t+2)) &= 0 \\ \sin(3\pi(t+2)) &= 0 \\ 3\pi(t+2) &= \pi k,\ k \in \mathbb{Z} \\ t+2 &= \frac{k}{3},\ k \in \mathbb{Z} \\ t &= -2 + \frac{k}{3},\ k \in \mathbb{Z}. \end{aligned}$$

3. For each function or equation, compute the derivative.

(a) $g(u) = \sqrt{2u} - \sqrt{5}u$

(b) $p(x) = e^{2x}\sin(2x)$

(c) $y = \sqrt{\sin(3x)}$

(d) $G(y) = \frac{2+\sin y}{2y+\cos y}$

(e) $y\sin(x^2) = x\sin(y^2)$

Solutions:

(a) Observe that we have two constant multiples: $g(u) = \sqrt{2}\cdot\sqrt{u} - \sqrt{5}\cdot u$. (Also, u is ***not*** in the second square root. ***Pay attention!*** *This was a WebAssign problem!)* Using the Constant Multiple and Power Rules,

$$g'(u) = \sqrt{2}\cdot\frac{1}{2}u^{-\frac{1}{2}} - \sqrt{5}\cdot 1 = \frac{\sqrt{2}}{2\sqrt{u}} - \sqrt{5}.$$

(b) Observe that $p(x)$ is a composition of functions: $p(u) = e^u \sin u$ where $u = 2x$. By the Chain Rule, $p'(x) = p'(u)\cdot u'$.
- To compute $p'(u)$, use the Product Rule and $p'(u) = e^u \sin u + e^u \cos u$.
- To compute u', use the Constant Multiple Rule and $u' = 2\cdot 1 = 2$.
- Thus $p'(x) = \left[e^{2x}\sin(2x) + e^{2x}\cos(2x)\right]\cdot 2$.

(c) Observe that $y(x)$ is a double composition of functions: $y(v) = \sqrt{v}$ where $v(u) = \sin u$ and $u(x) = 3x$. By the Chain Rule,

$$y'(x) = y'(v)\cdot v'(u)\cdot u'(x).$$

- To compute $y'(v)$, use the Power Rule and $y'(v) = \frac{1}{2}v^{-\frac{1}{2}} = \frac{1}{2\sqrt{v}}$.
- You should know that $v'(u) = \cos u$.
- To compute $u'(x)$, use the Constant Multiple Rule and $u'(x) = 3\cdot 1 = 3$.
- Thus

$$y'(x) = \frac{1}{2\sqrt{\sin(3x)}}\cdot\cos(3x)\cdot 3 = \frac{3\cos(3x)}{2\sqrt{\sin(3x)}}.$$

(d) To compute $G'(y)$, use the Quotient Rule and

$$\begin{aligned} G'(y) &= \frac{(0+\cos y)\cdot(2y+\cos y)-(2+\sin y)\cdot(2-\sin y)}{(2y+\cos y)^2} \\ &= \frac{2y\cos y+\cancel{\cos^2 y}-4+\cancel{\sin^2 y}^{\,1}}{(2y+\cos y)^2} \\ &= \frac{2y\cos y-3}{(2y+\cos y)^2}. \end{aligned}$$

Remark. We do *not* need implicit differentiation here. G is a function of y, so the derivative is with respect to y, not with respect to x. Thus $\frac{dy}{dx}$ does not need to appear in the derivative.

(e) To compute y', we need implicit differentiation. (Here y depends on x.) So

$$\frac{d}{dx}\left[y\sin\left(x^2\right)=x\sin\left(y^2\right)\right]$$

Both sides: Product Rule!

Left hand side: Chain Rule!	Right hand side: Chain Rule!
Let $u=x^2$;	Let $u=y^2$;
then $u'=2x$.	then $u'=2y\cdot y'$.

$$\begin{aligned} y'\cdot\sin\left(x^2\right)+y\cdot\cos\left(x^2\right)\cdot 2x &= 1\cdot\sin\left(y^2\right)+x\cdot\cos\left(y^2\right)\cdot 2y\cdot y' \\ y'\left[\sin\left(x^2\right)-2xy\cos\left(y^2\right)\right] &= \sin\left(y^2\right)-2xy\cos\left(x^2\right) \\ y' &= \frac{\sin\left(y^2\right)-2xy\cos\left(x^2\right)}{\sin\left(x^2\right)-2xy\cos\left(y^2\right)}. \end{aligned}$$

5. Differentiate

$$y=\left(x^2+1\right)^{\arctan x}.$$

Hint: Use logarithmic differentiation.

Solution:

Using properties of logarithms, we have

$$\begin{aligned} \ln y &= \ln\left(\left(x^2+1\right)^{\arctan x}\right) \\ \ln y &= \arctan x\cdot\ln\left(x^2+1\right). \end{aligned}$$

Now differentiate both sides with respect to x, and we have (dont' forget the Product and Chain Rules!)

$$\begin{aligned}
\frac{1}{y} \cdot y' &= \frac{1}{x^2+1} \cdot \ln\left(x^2+1\right) + \arctan x \cdot \frac{1}{x^2+1} \cdot 2x \\
\frac{1}{y} \cdot y' &= \frac{1}{x^2+1} \cdot \left(\ln\left(x^2+1\right) + 2x \arctan x\right) \\
y' &= \left(x^2+1\right)^{\arctan x} \cdot \frac{1}{x^2+1} \cdot \left(\ln\left(x^2+1\right) + 2x \arctan x\right) \\
y' &= \left(x^2+1\right)^{\arctan x - 1} \cdot \left(\ln\left(x^2+1\right) + 2x \arctan x\right).
\end{aligned}$$