MAT 167 TEST 2 FORM B (APPLICATIONS OF DERIVATIVES)

Directions: Solve each required problem **on a separate sheet of paper. Use pencil and show all work;** I deduct points for using pen or skipping important steps. **You must shut off your cell phone.** Some problems are worth more than others. Take your time; **quality is preferred to quantity.** I encourage you to ask questions.

1. Give (a) an intuitive definition of a critical point, and (b) a precise definition of a critical point.

Solution: A differentiable function f has a critical point at $x = a$ if
(a) f has a maximum, a minimum, or a horizontal tangent line at $x = a$.
(b) f' is zero or undefined at $x = a$.

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2. Find the radius and height of a cylindrical soda can with a volume of 500 cubic centimeters that minimizes the surface area.

Solution: We want to minimize $SA = 2\pi r^2 + 2\pi r h$. (The surface area is the sum of the areas of the top, the bottom and the side. The top and bottom are circles, with an area of πr^2 each. The side can be sliced vertically and unrolled to obtain a rectangle, whose vertical length is h, and whose horizontal length is the circumference of the circles, $2\pi r$.)

To minimize, we need to differentiate SA with respect to either r or h, preferably not both. We know that $V = 500$ and $V = \pi r^2 h$, so we can solve $h = \frac{500}{\pi r^2}$. (We know $V = \pi r^2 h$ because a cylinder is a right prism. Right prisms have volume $V = Bh$ where B is the area of the base. The base of a cylinder is a circle.)

Substitute this expression of h in terms of r into V to obtain

$$SA = 2\pi r^2 + 2\pi r \cdot \left(\frac{500}{\pi r^2}\right) = 2\pi r^2 + \frac{1000}{r}.$$

To minimize, we differentiate, set to zero, and solve for r:

$$\begin{aligned}\frac{dSA}{dr} &= 4\pi r - \frac{1000}{r^2}\\ 0 &= 4\pi r - \frac{1000}{r^2}\\ 4\pi r^3 &= 1000\\ r &= \sqrt[3]{\frac{250}{\pi}} \approx 4.3 \text{ cm}.\end{aligned}$$

Back-substitute to find h:

$$h = \frac{500}{\pi \sqrt[3]{\frac{250}{\pi}}^2} = \frac{2 \cdot \sqrt[3]{250}^3}{\sqrt[3]{\pi}^3 \cdot \frac{\sqrt[3]{250}^2}{\sqrt[3]{\pi}^2}} = \frac{2 \cdot \sqrt[3]{250}}{\sqrt[3]{\pi}} \approx 8.6\,\text{cm}.$$

Notice that $h = 2r$; that is, the height and the diameter are the same. Oil cans used to look like this, and some groceries are distributed with these dimensions. Soda is not among them.

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3. At what rate is soda being sucked out of a cylindrical glass that is 6 in tall and has a radius of 2 in? The depth of the soda decreases at a constant rate of 0.25 in/s.

Solution: The rate of change of soda is a rate of change of *volume*. The glass is cylindrical, so $V = \pi r^2 h$, where h represents the depth of the soda. Since the radius is constant regardless of the height of the soda, we can susbtitute $r = 2$ immediately, obtaining $V = 4\pi h$. Differentiate with respect to time and substitute $dh/dt = -0.25$ to find

$$\begin{aligned}\frac{dV}{dt} &= 4\pi \cdot \frac{dh}{dt} \\ &= 4\pi \cdot (-0.25) \\ &= -\pi.\end{aligned}$$

The soda is being sucked out (negative!) at roughly $\pi \approx 3.14\ \text{in}^3/\text{s}$.

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4. Let $f(x) = xe^{-x/2}$ and $I = [0,5]$.
   (a) Find the critical points of f on I.
   (b) Determine the absolute extreme values of f on I, when they exist.

Solution:

(a) To find the critical points, we compute the derivative, being sure to use the product and chain rules:

$$f'(x) = e^{-x/2} + xe^{-x/2} \cdot \left(-\frac{1}{2}\right) = e^{-x/2}\left(1 - \frac{x}{2}\right).$$

This function is defined for all real values of x, and powers of e are never zero, so critical points can occur only when

$$1 - \frac{x}{2} = 0 \quad \Longrightarrow \quad x = 2.$$

(b) We find the absolute extrema by checking both the critical points and the endpoints:

$$\begin{aligned}f(0) &= 0 \\ f(2) &= 2e^{-1} \approx 0.7 \\ f(5) &= 5e^{-5/2} \approx 0.4.\end{aligned}$$

The absolute minimum is 0, which occurs at $x = 0$. The absolute maximum is $2/e$, which occurs at $x = 2$.

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5. **[corrected]** The profit function of selling x widgets is $P(x) = -.01x^2 + 5.06x + 9359.91$, where $0 \le x \le 1000$.
   (a) Find the average profit function.
   (b) Find the marginal profit function.
   (c) Determine the average and marginal profits when $x = 10$.

(d) At what point is profit *maximized?*

Solution:

(a) Average profit is

$$\overline{P}(x) = \frac{P(x)}{x} = \frac{-.01x^2 + 5.06x + 9359.91}{x} = -.01x + 5.06 + \frac{9359.91}{x}.$$

(b) Marginal profit is

$$P'(x) = -.02x + 5.06.$$

(c) We have

$$\overline{P}(10) = -.1 + 5.06 + 935.991 = 940.951$$
$$P'(10) = -.2 + 5.06 = 4.86.$$

These are in dollars, of course.

(d) Profit is maximized either at an endpoint, or at a critical point. This function is a downward-pointing parabola, so it is maximized at the vertex, which is a critical point (horizontal slope!). This occurs when

$$\begin{aligned} 0 &= P'(x) \\ &= -.02x + 5.06 \\ x &= 2.53. \end{aligned}$$

That is, profit is maximized when selling roughly two and a half widgets. Good luck with that.