MAT 167 TEST 2 FORM B (DERIVATIVES)

Directions: Solve each required problem **on a separate sheet of paper. Use pencil and show all work;** I deduct points for using pen or skipping important steps. **You must shut off your cell phone.** Some problems are worth more than others. Take your time; **quality is preferred to quantity.** I encourage you to ask questions.

1. Give (a) an intuitive explanation as to why the derivative of the function $f(x) = x$ at a point $x = a$ is 1, and (b) a precise explanation as to why the derivative of f at $x = a$ is 1.

Solution: The derivative of the function $f(x) = x$ at $x = a$ is 1 because

(a) f is a line, so the line tangent to it is the same line; since f has slope 1, so will the tangent line, and the derivative is the slope of the tangent line.

(b) by definition,

$$f'(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x-a}{x-a} = \lim_{x\to a} 1 = 1.$$

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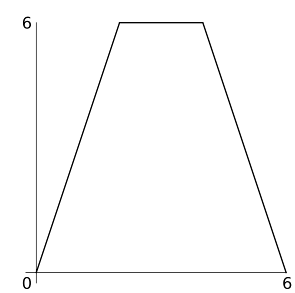
2. The graph of $f(x)$ is shown at [upper] right.
   (a) Sketch a graph of $f'(x)$.
   (b) At what values of x is $f(x)$ not continuous?
   (c) At what values of x is $f(x)$ not differentiable?

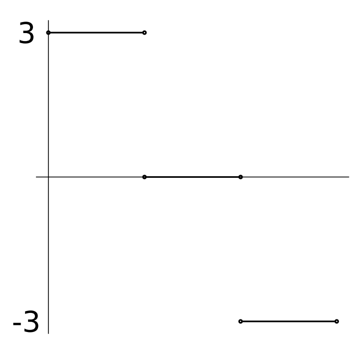


Solution:

(a) The graph I had is shown at lower right, and is made using the derivative of the function I actually used. You probably can't see them unless you enlarge the image, but those are holes at the ends of the line segments. I accepted some different graphs, as well. What mattered was that the value of the derivative at any point x had to look like it was the slope of the line tangent to the original function at that point: when the original graph is rising (resp. declining) linearly, the derivative should have a constant value of 3 (resp. -3), and when the original graph is horizontal, the derivative should have a constant value of 0.

(b) It looks as if f is continuous on its domain $[0, 6]$.

(c) It looks as if f is continuous on its domain $[0, 6]$ *except* at the points where the deirvative changes suddenly, $x = 2$ and $x = 4$.



3. Compute the derivatives of the following functions. You may use any property or shortcut we have described in class.

(a) 7^x	(b) $3\sqrt{x} - 4x^2 + 2$	(c) 3^5
(d) $\tan x \left(\sin x - x^2\right)$	(e) $\dfrac{e^x}{x^2}$	(f) $(2x-1)^5$
(g) $\cot^2 x$	(h) $(\arcsin(2x) - 3)^{25}$	(i) $\ln(3x-5)$

Solution:

(a) $7^x \ln 7$

(b) $3 \cdot \frac{1}{2} x^{-\frac{1}{2}} - 8x + 0 = \frac{3}{2\sqrt{x}} - 8x$

(c) 0, since 3^5 is a constant

(d) use the product rule: $\sec^2 x \left(\sin x - x^2\right) + \tan x (\cos x - 2x)$

(e) use the quotient rule:

$$\frac{e^x \cdot x^2 - e^x \cdot 2x}{\left(x^2\right)^2} = \frac{xe^x (x-2)}{x^4} = \frac{e^x (x-2)}{x^3}$$

(f) use the chain rule: $5(2x-1)^4 \cdot 2 = 10(2x-1)^4$

(g) use the chain rule: $2 \cot x \cdot \left(-\csc^2 x\right)$

(h) use the chain rule, *twice:*

$$25 (\arcsin(2x) - 3)^{24} \cdot \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$$

(i) use the chain rule:

$$\frac{1}{3x-5} \cdot 3$$

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4. Where does the equation $f(x) = e^x / x^2$ have a horizontal tangent line?

Solution: We need the derivative, which is the slope of the tangent line, to be zero. Since $f'(x) = e^x (x-2)/x^3$ (see 3(e)) we have

$$\frac{e^x (x-2)}{x^3} = 0$$
$$e^x (x-2) = 0.$$

Since $e^x \neq 0$ for any value of x, we solve

$$x - 2 = 0$$
$$x = 2.$$

Hence, f has a horizontal tangent line when $x = 2$.

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5. Use implicit differentation to compute dy/dx at $(\pi, 0)$ if $e^{x+y} = xy$.

Solution: Differentiation, with the chain rule and principles of implicit differentiation, gives us

$$\begin{aligned} e^{x+y} \cdot (1+y') &= y + xy' \\ e^{x+y} + y'e^{x+y} &= y + xy' \\ y'\left(e^{x+y} - x\right) &= y - e^{x+y} \\ y' &= \frac{y - e^{x+y}}{e^{x+y} - x}. \end{aligned}$$

Unfortunately, when I changed the equation from Form A, I neglected to change the point as well, and the point $(\pi, 0)$ does *not* lie on this curve. If it *were* on the curve, we would find dy/dx at $(\pi, 0)$ by substitution:

$$y' = \frac{0 - e^{\pi+0}}{e^{\pi+0} - \pi} = \frac{-e^{\pi}}{e^{\pi} - \pi}.$$

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6. Compute the derivative of

$$\frac{\tan^5(3x-2)\sin^3(2x)}{x^{12}(4x+1)^3}.$$

Solution: It's a lot easier if you use logarithmic differentiation, which was the point...

$$\begin{aligned} y &= \frac{\tan^5(3x-2)\sin^3(2x)}{x^{12}(4x+1)^3} \\ \ln y &= \ln \frac{\tan^5(3x-2)\sin^3(2x)}{x^{12}(4x+1)^3} \\ &= 5\ln\tan(3x-2) + 3\ln\sin(2x) - 12\ln x - 3\ln(4x+1). \end{aligned}$$

Now differentiate, preferably without neglecting the chain rule or principles of implicit differentiation:

$$\begin{aligned} \frac{1}{y} \cdot y' &= 5 \cdot \frac{1}{\tan(3x-2)} \cdot \sec^2(3x-2) \cdot 3 + 3 \cdot \frac{1}{\cos(2x)} \cdot 2 - 12 \cdot \frac{1}{x} - 3 \cdot \frac{1}{4x+1} \cdot 4 \\ \frac{y'}{y} &= \frac{15\sec^2(3x-2)}{\tan(3x-2)} + \frac{6}{\cos(2x)} - \frac{12}{x} - \frac{12}{4x+1} \\ y' &= \underbrace{\frac{\tan^5(3x-2)\sin^3(2x)}{x^{12}(4x+1)^3}}_{y} \cdot \left[\frac{15\sec^2(3x-2)}{\tan(3x-2)} + \frac{6}{\cos(2x)} - \frac{12}{x} - \frac{12}{4x+1}\right]. \end{aligned}$$
