MAT 167 TEST 2 FORM A (DERIVATIVES)

Directions: Solve each required problem **on a separate sheet of paper. Use pencil and show all work;** I deduct points for using pen or skipping important steps. **You must shut off your cell phone.** Some problems are worth more than others. Take your time; **quality is preferred to quantity.** I encourage you to ask questions.

1. Give (a) an intuitive explanation as to why the derivative of the function $f(x) = c$ at a point $x = a$ is 0, and (b) a precise explanation as to why the derivative of f at $x = a$ is 0.

Solution: The derivative of the function $f(x) = c$ at $x = a$ is 0 because

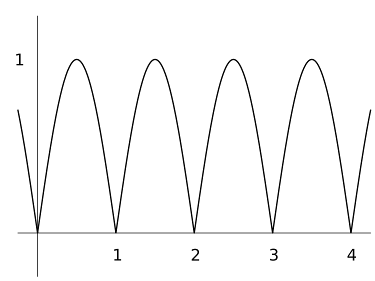
(a) f is a line, so the line tangent to it is the same line; since f has zero slope, so will the tangent line, and the derivative is the slope of the tangent line.

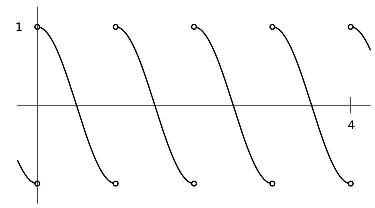
(b) by definition,

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{c - c}{x - a} = \lim_{x \to a} 0 = 0.$$

---

2. The graph of $f(x)$ is shown at [upper] right.
   (a) Sketch a graph of $f'(x)$.
   (b) At what values of x is $f(x)$ not continuous?
   (c) At what values of x is $f(x)$ not differentiable?





Solution:

(a) The graph I had is shown at lower right, and is made using the derivative of the function I actually used. I accepted some different graphs, as well. What mattered was that the value of the derivative at any point x had to look like it was the slope of the line tangent to the original function at that point: when the original graph seems to level off at its peaks, the derivative should intersect the x-axis, and when the original graph "bounces", the derivative should make a sudden jump (or have an asymptote).

(b) If f continues indefinitely as it does in the graph, it is always continuous.

(c) If f continues indefinitely as it does in the graph, it is non-differentiable at integer values of x, because it has a cusp at those points.

---

3. Compute the derivatives of the following functions. You may use any property or shortcut we have described in class.

(a) x^7	(b) 2^π	(c) $4\sqrt{x} - 3x^2 + 2$
(d) $x^2 e^x$	(e) $\dfrac{\tan x}{\sin x + 1}$	(f) 3^{2x-1}
(g) $(\arctan(2x) - 3)^{25}$	(h) $\sec^2 x$	(i) $\ln(5x - 3)$

Solution:

(a) $6x^7$

(b) 0, since 2^π is a constant

(c) $4 \cdot \frac{1}{2} x^{-\frac{1}{2}} - 6x + 0 = 2/\sqrt{x} - 6x$

(d) use the product rule: $2xe^x + x^2 e^x$

(e) use the quotient rule:

$$\frac{\sec^2 x \cdot (\sin x + 1) - \tan x \cdot \cos x}{(\sin x + 1)^2} = \frac{\sec^2 x \cdot (\sin x + 1) - \sin x}{(\sin x + 1)^2}$$

(f) use the chain rule: $3^{2x-1} \ln 3 \cdot 2$

(g) use the chain rule, *twice:*

$$25(\arctan(2x) - 3)^{24} \cdot \frac{1}{(2x)^2 + 1} \cdot 2$$

(h) use the chain rule: $2 \sec x \cdot \sec x \tan x$

(i) use the chain rule:

$$\frac{1}{5x - 3} \cdot 5$$

---

4. Where does the equation $f(x) = x^2 e^x$ have a horizontal tangent line?

Solution: We need the derivative, which is the slope of the tangent line, to be zero. Since $f'(x) = 2xe^x + x^2 e^x$ (see 3(d)) we have

$$\begin{aligned} 2xe^x + x^2 e^x &= 0 \\ e^x \left(2x + x^2\right) &= 0. \end{aligned}$$

Since $e^x \neq 0$ for any value of x, we solve

$$\begin{aligned} 2x + x^2 &= 0 \\ x(2 + x) &= 0 \\ x &= 0, 2. \end{aligned}$$

Hence, f has a horizontal tangent line when $x = 0$ or $x = 2$.

---

5. Use implicit differentation to compute dy/dx at $(\pi, 0)$ if $\sin(x+y) = xy$.

Solution: Differentiation, with the chain rule and principles of implicit differentiation, gives us

$$\begin{aligned}
\cos(x+y)\cdot\left(1+y'\right) &= 1\cdot y + x\cdot y' \\
\cos(x+y) + y'\cos(x+y) &= y + xy' \\
y'\left[\cos(x+y) - x\right] &= y - \cos(x+y) \\
y' &= \frac{y-\cos(x+y)}{\left[\cos(x+y)-x\right]}.
\end{aligned}$$

To find dy/dx at $(\pi, 0)$, we substitute $x = \pi$ and $y = 0$ to obtain

$$y' = \frac{0-\cos(\pi+0)}{\cos(\pi+0)-\pi} = \frac{-1}{1-\pi} = \frac{1}{\pi-1}.$$

---

6. Compute the derivative of

$$\frac{(3x-2)^5\sin^7(2x)}{x^3(4x+1)^{12}}.$$

Solution: It's a lot easier if you use logarithmic differentiation, which was the point...

$$\begin{aligned}
y &= \frac{(3x-2)^5\sin^7(2x)}{x^3(4x+1)^{12}} \\
\ln y &= \ln\frac{(3x-2)^5\sin^7(2x)}{x^3(4x+1)^{12}} \\
\ln y &= 5\ln(3x-2) + 7\ln\sin(2x) - 3\ln x - 12\ln(4x+1).
\end{aligned}$$

Now differentiate, preferably without neglecting the chain rule or principles of implicit differentiation:

$$\begin{aligned}
\frac{1}{y}\cdot y' &= 5\cdot\frac{1}{3x-2}\cdot 3 + 7\frac{1}{\sin(2x)}\cdot\cos(2x)\cdot 2 - 3\cdot\frac{1}{x} - 12\cdot\frac{1}{4x+1}\cdot 4 \\
\frac{y'}{y} &= \frac{15}{3x-2} + 14\cot(2x) - \frac{3}{x} - \frac{48}{4x+1} \\
y' &= \underbrace{\frac{(3x-2)^5\sin^7(2x)}{x^3(4x+1)^{12}}}_{y}\cdot\left[\frac{15}{3x-2} + 14\cot(2x) - \frac{3}{x} - \frac{48}{4x+1}\right].
\end{aligned}$$
