MAT 167 TEST 1 FORM A (LIMITS)

Directions: Solve each required problem **on a separate sheet of paper. Use pencil and show all work;** I deduct points for using pen or skipping important steps. **You must shut off your cell phone.** Some problems are worth more than others. Take your time; **quality is preferred to quantity.** I encourage you to ask questions.

1. Give (a) an intuitive definition of the limit of a function f at a point $x = a$, and (b) the precise definition of the limit of a function f at a point $x = a$.

Solution: $\lim_{x \to a} f(x) = L$ if

(a) L is the y-value that f approaches as x approaches a.
(b) for any $\epsilon > 0$, we can find $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - L| < \epsilon$.

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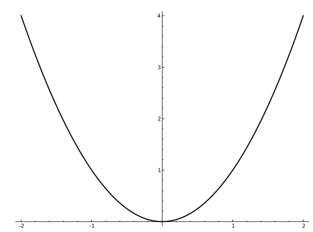
2. Graph the parabola $f(x) = x^2$. Explain why the secant lines between the points $(-a, f(-a))$ and $(a, f(a))$ have zero slope. What is the slope of the tangent line at $x = 0$, and *why?*

Solution: The graph of $f(x) = x^2$ is at right. The secant lines between $(-a, f(-a))$ and $(a, f(a))$ have slope

$$m_{\text{sec}} = \frac{f(-a) - f(a)}{-a - a} = \frac{a^2 - a^2}{-2a} = 0.$$

The slope of the tangent line at $x = 0$ is also zero, because

$$m_{\tan} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0.$$



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3. Use a table of values to estimate $\lim_{x \to 0^+} (x \ln x)$, correct to two decimal places.

Solution: I would use the table below, but there are other possibilities.

x	1	0.1	0.01	0.001	10^{-4}	10^{-5}	10^{-6}
$x \ln x$	0	-0.2303	-0.0461	-0.0069	-0.0009	10^{-4}	10^{-5}

(You don't have to use the 10^{-n} notation. I did so in order to save time.)

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4. If possible, determine a value of b such that $\lim_{x \to 2} p(x)$ exists.

$$p(x) = \begin{cases} 3x + b & \text{if } x \leq 2 \\ x - 2 & \text{if } x > 2. \end{cases}$$

Solution: We need $\lim_{x\to 2^-} p(x) = \lim_{x\to 2^+} p(x)$. For the left-sided limit, $x < 2$, so we use $p(x) = 3x + b$. For the right-sided limit, $x > 2$, so we use $p(x) = x - 2$. By substitution,

$$\begin{aligned} \lim_{x\to 2^-} (3x+b) &= \lim_{x\to 2^+} (x-2) \\ 6 + b &= 0 \\ b &= -6. \end{aligned}$$

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5. If $f(x) = 3x - 4$, find
   (a) $L = \lim_{x\to 3} f(x)$ using the properties of limits; and
   (b) δ such that if $|x-3| < \delta$, then $|f(x) - L| < 0.01$.

Solution: (a) Since f is a polynomial, we can substitute: $f(x) = 3 \cdot 3 - 4 = 5$. We can also use more basic properties:

$$\begin{aligned} \lim_{x\to 3} (3x-4) &= \lim_{x\to 3} (3x) - \lim_{x\to 3} 4 && \text{(sum property of limits)} \\ &= 3 \lim_{x\to 3} x - \lim_{x\to 3} 4 && \text{(constant multiple property)} \\ &= 3 \cdot 3 - \lim_{x\to 3} 4 && \text{(limit of powers of } x\text{)} \\ &= 9 - 4 && \text{(limit of a constant)} \\ &= 5. \end{aligned}$$

(b) We need

$$\begin{aligned} |f(x) - 5| &< 0.01 \\ |(3x-4) - 5| &< 0.01 \\ |3x - 9| &< 0.01 \\ 3|x-3| &< 0.01 \\ |x-3| &< 0.0\overline{3} = \frac{1}{30}. \end{aligned}$$

The left side has the form $|x-3| < \delta$! Thus, if we set $\delta = \frac{1}{30}$, then $|f(x) - L| < 0.01$.

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6. Determine the interval(s) on which the function $g(x) = \sqrt{x^4 - 1}$ is continuous.

Solution: We know from class that this type of function — the composition of two functions that are continuous on their domains — is continuous on its domain. So, we need merely find the domain. A number inside a square root should be nonnegative:

$$\begin{aligned} x^4 - 1 &\geq 0 \\ x^4 &\geq 1. \end{aligned}$$

This is true if $x \geq 1$ or if $x \leq 1$. Hence, g is continuous on $(-\infty, 1] \cup [1, \infty)$.

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7. Identify and classify (as holes, jumps, or vertical asymptotes) all the discontinuities of

$$r(x) = \frac{5x^2 - 15x}{2x^2 - 4x}.$$

Also find the horizontal asymptotes. Do not use any shortcuts you may have learned in pre-calculus, unless you're willing to explain them using principles of Calculus.

Solution: We know from class that this function — a *ratio*nal function — is continuous on its domain. It is define everywhere except where the denominator is zero. That is,

$$\begin{aligned} 2x^2 - 4x &= 0 \\ 2x(x-2) &= 0 \\ x &= 0, 2. \end{aligned}$$

Hence, it is discontinuous at $x = 0$ and at $x = 2$. When $x = 0$, the numerator is also zero, so we have a hole. When $x = 2$, the numerator is nonzero, so we have a vertical asymptote.

We find the horizontal asymptote by considering

$$\lim_{x\to\infty} r(x) = \lim_{x\to\infty} \frac{5x^2 - 15x}{2x^2 - 4x} = \lim_{x\to\infty} \left(\frac{5x^2 - 15x}{2x^2 - 4x} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \right) = \lim_{x\to\infty} \frac{5 - \cancelto{0}{\frac{15}{x}}}{2 - \cancelto{0}{\frac{4}{x}}} = \frac{5}{2}.$$

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8. List the first five terms of the sequence $\frac{n^2+1}{n}$, then find $\lim_{n\to\infty} \frac{n^2+1}{n}$.

Solution: The first five terms are

$$\frac{1+1}{1} = 2, \quad \frac{4+1}{2} = \frac{5}{2}, \quad \frac{9+1}{3} = \frac{10}{3}, \quad \frac{16+1}{4} = \frac{17}{4}, \quad \frac{25+1}{5} = \frac{26}{5}.$$

The limit is

$$\lim_{n\to\infty} \left(\frac{n^2+1}{n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \right) = \lim_{n\to\infty} \frac{n + \cancelto{0}{\frac{1}{n}}}{1} = \infty.$$

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9. The table below gives various values of three functions f, g, h around the point $x = \pi$.
   (a) What would be the value of $\lim_{x\to\pi} f(x)$ if we applied the Squeeze Theorem?
   (b) What aspects of the data suggest that we can apply the Squeeze Theorem?
   (c) What additional information would we need to be sure that we can apply the Squeeze Theorem?

x	2.5	3	3.1	$\lim_{x\to\pi} \cdots$	3.2	3.3	3.5
$g(x)$	11	7	6	5.8	5.5	5.3	5.1
$f(x)$	10	-5	6	?	5	-1	4
$h(x)$	-20	-10	0	5.8	5	-3	-12

Solution: (a) 5.8

(b) It *looks* as if $h(x) \le f(x) \le g(x)$ for all x in the domain $[2.5, 3.5]$.

(c) Since the table only gives a sample of six points out of infinitely many, we would need to know that f really does lie between g and h for all $x \in [2.5, 3.5]$ — or at least for some neighborhood of π.