MAT 102 TEST 3, SOLUTIONS

Most solutions are for Form A. Solutions for Form B are usually obtained by modifying the numbers appropriately. When this is not the case, or is not so obvious, I have noted the different answer for Form B.

If you think you find an error, let me know. If you are right, I will give you extra credit.

1. (Form A) Give the basic formula for **profit**.
   The basic formula for **profit** is $P(x) = R(x) - C(x)$, where R is **revenue** and C is **total cost**.
   (Form B) What is the difference between an **absolute maximum** and a **relative maximum**?
   An **absolute maximum** is the largest y-value over the entire domain. A **relative maximum** is the largest y-value only in a neighborhood around the point.

2. Suppose the average fuel economy of a particular car is $E(x) = -.01x^2 + 0.8x + 7.3$, where x is the driving speed in miles per hour ($20 \le x \le 60$). Use Calculus to determine the speed at which fuel economy is greatest.
   We need to take the derivative and solve for the root(s). The derivative is $E'(x) = -.02x + 0.8$, and the root occurs at $x = 40$. Fuel economy is greatest at 40 miles per gallon.
   The answer on Form B is slightly different, but is found the same way.

3. Find x (*without* rounding) if

$$\text{(a) } 3^x = 1/27 \quad \text{(b) } \ln x = 0 \quad \text{(c) } \log_5 x = 3$$

   (a) $x = -3$
   (on Form B, which has $4^x = 1/16$, the answer is $x = -2$)
   (b) $x = e^0 = 1$ (this is an "obvious" simplification, so you must perform it)
   (on Form B, which has $\ln x = 1$, the answer is $x = e^1 = e$)
   (c) $x = 5^3 = 125$
   (on Form B, which has $\log_2 x = 3$, the answer is $x = 2^3 = 8$)

4. If an investment is compounded monthly at 4.5%, how long will it take to double the principal?
   The formula for compound interest is $A = P(1 + r/n)^{nt}$ where A is the value of the principal P after t years when compounded n times a year. We want to know how long it takes for the principal to double, which occurs when $A = 2P$. We solve

$$2\cancel{P} = \cancel{P}\left(1 + \frac{.045}{12}\right)^{12t}$$

$$2 = \left(1 + \frac{.045}{12}\right)^{12t}$$

$$\ln 2 = \ln\left(1+\frac{.045}{12}\right)^{12t}$$
$$\ln 2 = (12t)\ln\left(1+\frac{.045}{12}\right)$$
$$\frac{\ln 2}{12\ln\left(1+\frac{.045}{12}\right)} = t$$

This approximates to 15.43 years, which we round up to 16 years. (The rule of 70 estimates $70/4.5 \approx 15.55$ years.)

(On Form B, which has a rate of 3.5%, the answer is 19.833 years, which we round up to 20 years. The rule of 70 estimates $70/3.5 \approx 20$ years.)

5. Find the derivatives of

$$\text{(a) } x^7\ln\left(x^3+2\right) \quad \text{(b) } \frac{e^{2x}}{x^3} \quad \text{(c) } \sqrt{t^4-3\ln t}$$

(a) We need *both* the product rule *and* the chain rule.

$$\underbrace{7x^6}_{\frac{d}{dx}\text{first}}\underbrace{\ln\left(x^3+2\right)}_{\text{second}}+\underbrace{x^7}_{\text{first}}\cdot\underbrace{\underbrace{\frac{1}{x^3+2}}_{\frac{d}{dx}\ln u}\cdot\underbrace{\left(3x^2+0\right)}_{\frac{du}{dx}}}_{\frac{d}{dx}\text{second}}=7x^6\ln\left(x^3+2\right)+\frac{3x^9}{x^3+2}.$$

(b) We need *both* the quotient rule *and* the chain rule.

$$\frac{\underbrace{\underbrace{e^{2x}}_{\frac{d}{dx}e^u}\cdot\underbrace{2}_{\frac{du}{dx}}}_{\frac{d}{dx}\text{first}}\cdot\underbrace{x^3}_{\text{second}}-\underbrace{e^{2x}}_{\text{first}}\cdot\underbrace{3x^2}_{\frac{d}{dx}\text{second}}}{\left(x^3\right)^2}=\frac{2x^3e^{2x}-3x^2e^{2x}}{x^6}=\frac{\cancel{x^2}^{\,1}\left(2xe^{2x}-3e^{2x}\right)}{\cancel{x^6}_{\,x^4}}=\frac{2xe^{2x}-3e^{2x}}{x^4}.$$

I would give full credit if you only had the first fraction.

(c) We need the chain rule again. Remember that a square root is the same as a $1/2$ power.

$$\underbrace{\frac{1}{2}\left(t^4-3\ln t\right)^{-\frac{1}{2}}}_{\frac{d}{du}u^{\frac{1}{2}}}\cdot\underbrace{\left(4t^3-3\cdot\frac{1}{t}\right)}_{\frac{du}{dx}}=\frac{4t^3-\frac{3}{t}}{2\sqrt{t^4-3\ln t}}\cdot\underbrace{\frac{t}{t}}_{\text{b/c of } 3/t}=\frac{4t^4-3}{2t\sqrt{t^4-3\ln t}}.$$

6. Suppose the price function for a widget is $p(x) = 350e^{-0.1x}+20$, the fixed costs are \$50, and the variable costs are $v(x) = 50+20(x-1)$.

(a) Determine the revenue function.

Revenue is the product of price and number of units sold; that is, $R(x) = x \cdot p(x)$. So $R(x) = 350xe^{-0.1x}+20x$.

(b) Determine the profit function.

Profit, as noted above, is $R(x) - C(x)$. We just found R, and C is the sum of variable and fixed costs, so $C(x) = 50 + [50 + 20(x-1)]$, or more simply $C(x) = 100 + 20(x-1)$. Hence the profit function is $P(x) = \left(350xe^{-0.1x} + 20x\right) - [100 + 20(x-1)]$.

(c) Find the value of x that maximizes profit.

We need to solve for where the derivative equals zero. The derivative is

$$P'(x) = \left(\underbrace{\underbrace{350 \cdot 1}_{\frac{d}{dx}\text{first}} \cdot \underbrace{e^{-0.1x}}_{\text{second}} + \underbrace{350x}_{\text{first}} \cdot \underbrace{\underbrace{e^{-0.1x}}_{\frac{d}{du}e^u} \cdot \underbrace{(-0.1)}_{\frac{du}{dx}}}_{\frac{d}{dx}\text{second}}}_{\text{product rule}} + 20 \right) - [0 + 20(1-0)]$$

which simplifies to

$$P'(x) = 350e^{-0.1x} - 35xe^{-0.1x}.$$

Set this to zero and solve, to get

$$350e^{-0.1x} - 35xe^{-0.1x} = 0$$
$$35e^{-0.1x}(10 - x) = 0.$$

By the zero product rule, $e^{-0.1x} = 0$ or $10 - x = 0$. *No* power of e is *ever* equal to zero, so the first has no solution. The second gives us $x = 10$. So the value of x that maximizes profit is 10 widgets.

(d) Find the relative rate of change of profit at that value of x.

Relative rate of change is the ratio of derivative to quantity, so we want $P'(x)/P(x)$ at the value $x = 10$. However, we found $x = 10$ by setting $P'(x) = 0$, so we know already the result will be $0/\text{something} = 0$. If you disbelieve me, go ahead and do the substitution:

$$\frac{P'(10)}{P(10)} = \frac{350e^{-0.1\times 10} - 35 \cdot 10 \cdot e^{-0.1\times 10}}{(350 \cdot 10 \cdot e^{-0.1\times 10} + 20) - [100 + 20(10-1)]} = \frac{0}{\text{I don't really care, since the top is 0}} = 0.$$