MAT 102 TEST 1, FORM A

Directions: Solve these problems. You may write on this paper, but I will not read it. Problems are not weighted equally. Show all necessary work: **computations that are not obvious must be shown**. As for what is "obvious", better safe than sorry!

1. Give both a geometric definition and the precise definition of **instantaneous rate of change**.
   Geometrically, the instantaneous rate of change of a function f at $x=a$ is the slope of the line tangent to f at $x=a$. Precisely, it is

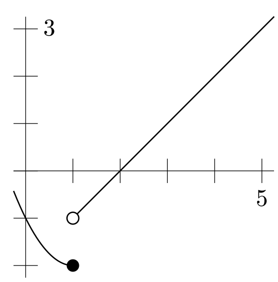
$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x},\quad \text{or,}\quad \lim_{x\to a}\frac{f(x)-f(a)}{x-a},\quad \text{or,}\quad \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

2. Compute $\lim_{x\to 3}\frac{x^2-9}{x-3}$ using a table of values.

x	y
2	5
2.5	5.5
2.9	5.9
2.99	5.99
3	?
3.01	6.01
3.1	6.1
3.5	6.5
4	7

   It looks as if the limit is 6.

3. Given the graph of $f(x)$ at right,
   (a) find $\lim_{x\to 1^-} f(x)$; -2
   (b) find $\lim_{x\to 1^+} f(x)$; -1
   (c) find $\lim_{x\to 1} f(x)$. DNE
   (d) Is f continuous at $x=1$? no



4. Let $f(x)=x^3-2$.
   (a) Find the average rate of change between $x=1$ and $x=2$.

$$\frac{\Delta y}{\Delta x}=\frac{f(2)-f(1)}{2-1}=\frac{(2^3-2)-(1^3-2)}{1}=\frac{6+1}{1}=7$$

(b) Find $f'(x)$ using the ***definition*** of the derivative.

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\left[(x+h)^3-2\right]-\left(x^3-2\right)}{h} \\ &= \lim_{h\to 0} \frac{\left[\left(\cancel{x^3}+3x^2h+3xh^2+h^3\right)-\cancel{2}\right]-\left(\cancel{x^3}-\cancel{2}\right)}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}\left(3x^2+3xh+h^2\right)}{\cancel{h}} \\ &= 3\cdot x^2+3x\cdot 0+0^2 \\ &= 3x^2. \end{aligned}$$

(c) Use your answer to part (b) to find the instantaneous rate of change at $x=1$.
By substitution, $f'(1)=3\cdot 1^2=3$.

5. Use the ***properties*** of the derivative to evaluate the derivative of each of the following functions.

(a) $f(x)=x^3-2$ (b) $f(x)=\left(x^3-2\right)\left(2x^2-x\right)$

(c) $f(x)=\dfrac{x^3-2}{2x^2-x}$ (d) $f(x)=3\left(x^3-2\right)^{20}-2$

(a) is straightforward: $f'(x)=3x^2-0=3x^2$.
(b) can be attacked by expansion or the product rule, which gives $3x^2\left(2x^2-x\right)+\left(x^3-2\right)\cdot(4x-1)$.
(c) requires the quotient rule:

$$f'(x)=\frac{\underbrace{3x^2}_{\text{deriv first}}\underbrace{\left(2x^2-x\right)}_{\text{second}}-\underbrace{\left(x^3-2\right)}_{\text{first}}\cdot\underbrace{4x-1}_{\text{deriv second}}}{\underbrace{\left(2x^2-x\right)^2}_{\text{second squared}}},$$

and you can stop there. In fact, you probably *should* stop there.
(d) requires the chain rule, because it has the form $y=3u^{20}-2$ where $u=x^3-2$. The chain rule is

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\left(3\cdot 20u^{19}-0\right)\cdot 3x^2=60u^{19}\cdot 3x^2=180x^2\left(x^3-2\right)^{19}.$$

6. Suppose the number of traffic fatalities per hundred million miles traveled approximates

$$f(x)=\frac{2}{\sqrt{x}}+1,$$

where x stands for the number of years *since 1975*. Find the (instantaneous) rate of change of this percentage in the year 2006, and interpret your answer.

The year 2006 occurs when $x = 31$ (years *since 1975*) and "rate of change" means "derivative". So,

$$f'(x) = 2 \cdot \left(-\frac{1}{2} x^{-3/2}\right) + 0 = -\frac{1}{x\sqrt{x}}.$$

Evaluated at $x = 31$, we find that

$$f'(31) = -\frac{1}{31\sqrt{31}} \approx -0.00579.$$

This means that the number of traffic fatalities was decreasing by roughly .6% in 2006.