MAT 102 TEST "4", SOLUTIONS

I don't ordinarily give a test on Chapter 5; it's rolled into the final exam. This gives you an idea of what sorts of questions I might ask on the exam **for Chapter 5 material only**.

If you think you find an error, let me know. If you are right, I will give you extra credit.

1. Define the following terms, and/or explain the following theorems.
   (a) the definite integral $\int_a^b f(x)\,dx$ (geometric and precise definitions)
   geometric: the area between $f(x)$ and the x-axis on the interval $[a,b]$.
   precise: $\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n}(f(x)\Delta x)\right]$ where $\Delta x = \frac{b-a}{n}$. I won't require that you write the limits on Σ or the definition of Δx.
   (b) the Fundamental Theorem of Calculus (in words and in symbols)
   in words: the area between $f(x)$ and the x-axis on the interval $[a,b]$ is $F(b)-F(a)$, where F is any antiderivative of f
   in symbols: $\int_a^b f(x)\,dx = F(b)-F(a)$, where $F(x)=\int f(x)\,dx$

2. Find the area between the curves $f(x)=x^2+4$, $g(x)=-2x$, between $x=0$ and $x=3$. (These curves do not intersect.)
   To decide which curve is on top, evalue the y-values at an x-value in the interval:
   $$f(1)=5>-2=g(1),$$
   so f is on top. To find the area, then, we compute
   $$\begin{aligned}\int_0^3 f(x)-g(x)\,dx &= \int_0^3 (x^2+4)-(-2x)\,dx\\ &= \int_0^3 x^2+2x+4\,dx\\ &= \frac{x^3}{3}+\cancel{2}\cdot\frac{x^2}{\cancel{2}}+4x\Big|_0^3\\ &= \left(\frac{3^3}{3}+3^2+4\cdot 3\right)-\left(\frac{0^3}{3}+0^2+4\cdot 0\right)\\ &= 30.\end{aligned}$$

3. Compute the following integrals. Part (c) requires substitution.
   (a) $\int 6e^{3x}\,dx$
   $$\int 6e^{3x}\,dx = 6\int e^{3x}\,dx = \underbrace{6\cdot\frac{e^{3x}}{3}+C}_{\text{full credit}} = 2e^{3x}+C$$

(b) $\int_1^3 x^2 - 3\,dx$

$$\int_1^3 x^2 - 3\,dx \underset{\text{FTC}}{=} \frac{x^3}{3} - 3x\Big|_1^3 = \left(\frac{3^3}{3} - 3\cdot 3\right) - \left(\frac{1^3}{3} - 3\cdot 1\right) = \frac{8}{3}.$$

(c) $\int \frac{3e^{3x}}{\sqrt{e^{3x}-4}}\,dx$

As noted, this requires substitution. The best approach is to look at the "inside" of the square root: let $u = e^{3x} - 4$; then $u' = e^{3x}\cdot 3$, and the integral now has the form

$$\int \frac{3e^{3x}}{\sqrt{e^{3x}-4}}\,dx = \int \frac{1}{\sqrt{e^{3x}-4}} \cdot 3e^{3x}\,dx = \int \frac{1}{\sqrt{u}} \cdot u'\,dx \underset{\text{Chain Rule}}{=} \int \frac{1}{\sqrt{u}}\,du.$$

This is straightforward:

$$\int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{u} + C = 2\sqrt{e^{3x}-4} + C.$$

4. In an effort to reduce its inventory, a photography store runs a sale on its least popular compact cameras. The sales rate (cameras sold per day) of the sale is predicted to be $4/t$, where $t = 1$ corresponds to the beginning of the sale, at which time none of the inventory of 45 cameras had been sold.

(a) Find a formula for the total number of cameras sold up to day t.

We are given the sales rate, $4/t$. A "rate" is a derivative, so the total number $S(t)$ of cameras sold is the antiderivate of the rate:

$$S(t) = \int \frac{4}{t}\,dt = 4\int \frac{1}{t}\,dt = 4\ln t + C.$$

We are also told that *no* cameras were sold at time $t = 1$, the beginning of the sale. (This should make sense.) That means $S(1) = 0$. We use this to find a precise value of C:

$$\begin{aligned} 0 = S(1) &= 4\ln 1 + C \\ 0 &= 4\cdot 0 + C \\ 0 &= C. \end{aligned}$$

Thus the formula for the total number of cameras sold up to day t is

$$S(t) = 4\ln t + 0.$$

(b) Will the store have sold its inventory of 45 cameras by day $t = 29$?

On day 29, the store will have sold

$$S(29) = 4\ln 29 + 0 \approx 13 \text{ cameras},$$

so no, the store will not have sold its inventory.

5. Find the average *value* of $f(x) = \frac{1}{x}$ on the interval from $x = 1$ to $x = 5$. Round your answer to three decimal places.
Don't confuse average *value* with average *rate of change;* I could ask for both! Average value is

$$\frac{1}{b-a} \cdot \int_a^b f(x)\,dx = \frac{1}{5-1} \cdot \int_1^5 \frac{1}{x}\,dx = \frac{1}{4}\left(\ln x|_1^5\right) = \frac{1}{4}(\ln 5 - \ln 1) = \frac{1}{4}(\ln 5 - 0).$$

Rounded to three decimal places, this is 0.402.