MAT 102 TEST 3, SOLUTIONS

Most solutions are for Form A. Solutions for Form B are usually obtained by modifying the numbers appropriately. When this is not the case, or is not so obvious, I have noted the different answer for Form B.

If you think you find an error, let me know. If you are right, I will give you extra credit.

1. Definitions omitted. I may not ask any on this test.

2. Find x (*without* rounding) if

   (a) $3^x = 27$ (b) $16^x = 1/64$ (c) $\ln x = 0$ (d) $\log_5 x = 3$ (e) $\log_3 27$ (f) $\log_{16} 1/64$

   (a) $x = 3$
   (b) Rewrite in base 4 as $\left(4^2\right)^x = 4^{-3}$, so that $2x = -3$ and $x = -3/2$. Or, rewrite in base 2 as $\left(2^4\right)^x = 2^{-6}$, so that $4x = -6$ and $x = -2/3$.
   (c) $x = e^0 = 1$ (this is an "obvious" simplification, so you must perform it)
   (d) $x = 5^3 = 125$
   (e) Rewrite $\log_3 27 = x$ in exponential form as $27 = 3^x$. We already solved this in part (a), so $x = 3$.
   (f) Rewrite $\log_{16} 1/64 = x$ in exponential form as $1/64 = 16^x$. We already solved this in part (b), so $x = -2/3$.

3. If an investment is compounded monthly at 4.5%, how long will it take to double the principal?

   The formula for compound interest is $A = P(1 + r/n)^{nt}$ where A is the value of the principal P after t years when compounded n times a year. We want to know how long it takes for the principal to double, which occurs when $A = 2P$. We solve

$$2\cancel{P} = \cancel{P}\left(1 + \frac{.045}{12}\right)^{12t}$$

$$2 = \left(1 + \frac{.045}{12}\right)^{12t}$$

$$\ln 2 = \ln\left(1 + \frac{.045}{12}\right)^{12t}$$

$$\ln 2 = (12t)\ln\left(1 + \frac{.045}{12}\right)$$

$$\frac{\ln 2}{12\ln\left(1 + \frac{.045}{12}\right)} = t$$

   This approximates to 15.43 years, which we round up to 16 years. (The rule of 70 estimates $70/4.5 \approx 15.55$ years.)

   (On Form B, which has a rate of 3.5%, the answer is 19.833 years, which we round up to 20 years. The rule of 70 estimates $70/3.5 \approx 20$ years.)

4. A newly-purchase piece of equipment is estimated to depreciate according to the function $v(t) = 16350e^{-0.25t}$, where v is the value in dollars of the equipment after t years.
   (a) What was the initial purchase price of the equipment?
   The initial purchase price is the same as the value of the equipment at time $t = 0$; that is, after 0 years. We find this using $v(0) = 16350e^{-0.25\times 0} = 16350e^0 = 16350 \times 1 = 16350$. The initial purchase price was \$16350.
   (b) How long will it take for the equipment to be worth 30% of its original value?
   Thirty percent of the original value is $.30 \times 16350 = 4905$. Substitute this into $v(t)$ to solve:

$$
\begin{aligned}
4905 &= 16350e^{-0.25t}\\
\frac{4905}{16350} &= e^{-0.25t}\\
\ln\left(\frac{4905}{16350}\right) &= \ln e^{-0.25t}\\
\ln\left(\frac{4905}{16350}\right) &= -0.25t\\
-\frac{\ln\left(\frac{4905}{16350}\right)}{0.25} &= t.
\end{aligned}
$$

   This approcimates to 4.82 years, which we round up to 5 years.

5. Find the derivatives of

$$(a)\ x^7 \ln\left(x^3+2\right) \quad (b)\ \frac{e^{2x}}{x^3} \quad (c)\ \sqrt{t^4 - 3\ln t}$$

   (a) We need *both* the product rule *and* the chain rule.

$$\underbrace{7x^6}_{\frac{d}{dx}\text{first}}\underbrace{\ln\left(x^3+2\right)}_{\text{second}} + \underbrace{x^7}_{\text{first}} \cdot \underbrace{\underbrace{\frac{1}{x^3+2}}_{\frac{d}{dx}\ln u} \cdot \underbrace{\left(3x^2+0\right)}_{\frac{du}{dx}}}_{\frac{d}{dx}\text{second}} = 7x^6\ln\left(x^3+2\right) + \frac{3x^9}{x^3+2}.$$

   (b) We need *both* the quotient rule *and* the chain rule.

$$\frac{\underbrace{\underbrace{e^{2x}}_{\frac{d}{dx}e^u} \cdot \underbrace{2}_{\frac{du}{dx}}}_{\frac{d}{dx}\text{first}} \cdot \underbrace{x^3}_{\text{second}} - \underbrace{e^{2x}}_{\text{first}} \cdot \underbrace{3x^2}_{\frac{d}{dx}\text{second}}}{\left(x^3\right)^2} = \frac{2x^3e^{2x} - 3x^2e^{2x}}{x^6} = \frac{\cancel{x^2}^{1}\left(2xe^{2x} - 3e^{2x}\right)}{\cancel{x^6}_{x^4}} = \frac{2xe^{2x} - 3e^{2x}}{x^4}.$$

   I would give full credit if you only had the first fraction.

(c) We need the chain rule again. Remember that a square root is the same as a $^1/_2$ power.

$$\underbrace{\frac{1}{2}\left(t^4-3\ln t\right)^{-\frac{1}{2}}}_{\frac{d}{du}u^{\frac{1}{2}}}\cdot\underbrace{\left(4t^3-3\cdot\frac{1}{t}\right)}_{\frac{du}{dx}}=\frac{4t^3-\frac{3}{t}}{2\sqrt{t^4-3\ln t}}\cdot\underbrace{\frac{t}{t}}_{\text{b/c of } 3/t}=\frac{4t^4-3}{2t\sqrt{t^4-3\ln t}}.$$

6. Suppose the price function for a widget is $p(x)=350e^{-0.1x}+20$, the fixed costs are \$50, and the variable costs are $v(x)=50+20(x-1)$.

(a) Determine the revenue function.
Revenue is the product of price and number of units sold; that is, $R(x)=x\cdot p(x)$. So $R(x)=350xe^{-0.1x}+20x$.

(b) Determine the profit function.
Profit, as noted above, is $R(x)-C(x)$. We just found R, and C is the sum of variable and fixed costs, so $C(x)=50+[50+20(x-1)]$, or more simply $C(x)=100+20(x-1)$. Hence the profit function is $P(x)=\left(350xe^{-0.1x}+20x\right)-[100+20(x-1)]$.

(c) Find the value of x that maximizes profit.
We need to solve for where the derivative equals zero. The derivative is

$$P'(x)=\left(\underbrace{\underbrace{350\cdot1}_{\frac{d}{dx}\text{first}}\cdot\underbrace{e^{-0.1x}}_{\text{second}}+\underbrace{350x}_{\text{first}}\cdot\underbrace{\underbrace{e^{-0.1x}}_{\frac{d}{du}e^u}\cdot\underbrace{(-0.1)}_{\frac{du}{dx}}}_{\frac{d}{dx}\text{second}}}_{\text{product rule}}+20\right)-[0+20(1-0)]$$

which simplifies to

$$P'(x)=350e^{-0.1x}-35xe^{-0.1x}.$$

Set this to zero and solve, to get

$$350e^{-0.1x}-35xe^{-0.1x}=0$$
$$35e^{-0.1x}(10-x)=0.$$

By the zero product rule, $e^{-0.1x}=0$ or $10-x=0$. *No* power of e is *ever* equal to zero, so the first has no solution. The second gives us $x=10$. So the value of x that maximizes profit is 10 widgets.

(d) Find the relative rate of change of profit at that value of x.
Relative rate of change is the ratio of derivative to quantity, so we want $^{P'(x)}/_{P(x)}$ at the value $x=10$. However, we found $x=10$ by setting $P'(x)=0$, so we know already the result will be $^0/_{\text{something}}=0$. If you disbelieve me, go ahead and do the substitution:

$$\frac{P'(10)}{P(10)}=\frac{350e^{-0.1\times10}-35\cdot10\cdot e^{-0.1\times10}}{(350\cdot10\cdot e^{-0.1\times10}+20)-[100+20(10-1)]}=\frac{0}{\text{I don't really care, since the top is 0}}=0.$$

7. A widget sales outfit wants to increase its revenue. Market research suggests the demand function for a widget is $D(p) = p^3 e^{-0.1p}$, so long as $30 < p$.

(a) Find the function representing the elasticity of demand.

Use the formula

$$E(p) = -\frac{pD'(p)}{D(p)} = -\frac{p \times \underbrace{\left[\underbrace{3p^2}_{\frac{d}{dp}\text{first}} \cdot \underbrace{e^{-0.1p}}_{\text{second}} + \underbrace{p^3}_{\text{first}} \cdot \underbrace{\left(\underbrace{e^{-0.1p}}_{\frac{d}{du}e^u} \cdot \underbrace{-0.1}_{\frac{du}{dp}}\right)}_{\frac{d}{dp}\text{second — use chain rule}}\right]}_{D'(p)\text{ — use product rule}}}{p^3 e^{-0.1p}}$$

$$= -\frac{p\left(3p^2 e^{-0.1p} - 0.1p^3 e^{-0.1p}\right)}{p^3 e^{-0.1p}}$$

$$= -\frac{\underbrace{p^3 e^{-0.1p}(3 - 0.1p)}_{\text{factored out common } p^2 e^{-0.1p}}}{p^3 e^{-0.1p}}$$

$$= -3 + 0.1p.$$

(b) Evaluate the elasticity of $p = 31$. Should the company raise prices, lower them, or neither?

$E(31) = -3 + 0.1(31) = 0.1$. Demand is inelastic, so the company can raise prices and increase revenue.

(c) Evaluate the elasticity of $p = 80$. Should the company raise prices, lower them, or neither?

$E(80) = -3 + 0.1(80) = 5$. Demand is elastic, so the company should lower prices to increase revenue.

(d) *(Extra Credit)* Find the value of p that maximizes revenue.

Maximal revenue occurs at unit elasticity, so we want the value of p that gives us $E(p) = 1$. By substitution,

$$1 = -3 + 0.1p \implies 4 = 0.1p \implies 40 = p.$$

Revenue is maximized at a price of \$40.