MAT 102 TEST 2, FORM A

If you think you find an error, let me know. If you are right, I will give you extra credit.

1. Definitions omitted.

2. Let $f(x) = {}^{2x^2}/_{x^2-4}$.
   (a) Find the horizontal asymptote(s) of f, if any.
   Compare the degrees of numerator and denominator. Since they are equal (both 2) we determine the horizontal asymptote by dividing the leading coefficients of numerator and denominator. Thus, $y = {}^2/_1$ is the horizontal asymptote.
   (b) Find the vertical asymptote(s) of f, if any.
   We set the denominator to zero, and solve for x. The denominator is zero if $x^2 - 4 = 0$. Factoring gives us $(x-2)(x+2) = 0$. By the zero product rule, $x - 2 = 0$ or $x + 2 = 0$. Solving for x tells us $x = 2$ or $x = -2$. So, the vertical asymptotes are $x = \pm 2$.
   (c) Use a sign diagram to identify the regions where f is increasing and the regions where f is decreasing.
   We need the derivative:

$$f'(x) = \frac{(4x)\left(x^2-4\right) - \left(2x^2\right)(2x-0)}{\left(x^2-4\right)^2} = \frac{4x^3 - 16x - 4x^3}{\left(x^2-4\right)^2} = \frac{-16x}{\left(x^2-4\right)^2}.$$

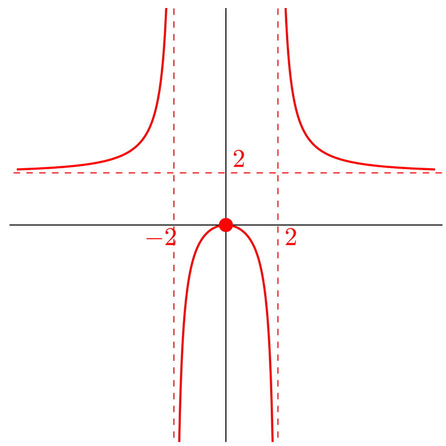
   Set it to zero, and we find that

$$\begin{aligned} \frac{-16x}{\left(x^2-4\right)^2} &= 0 \\ \left[\left(x^2-4\right)^2\right] \cdot \frac{-16x}{\left(x^2-4\right)^2} &= 0\left[\left(x^2-4\right)^2\right] \\ -16x &= 0 \\ x &= 0. \end{aligned}$$

   We have found only one critical point, $x = 0$. The sign diagram must check around the critical point *and the vertical asymptotes:*

$f'(x)$	$+$	und	$+$	0	$-$	und	$-$
x	-3	-2	-1	0	1	2	3

   The function is increasing on $(-\infty, -2) \cup (-2, -1)$ and decreasing on $(0, 2) \cup (2, \infty)$.
   (d) Identify the relative maxima and relative minima.
   From the sign diagram we infer there is a relative maximum *at* $x = 0$. Substituting this into the original function, we find the relative maximum *is* $f(0) = {}^{2(0)}/_{(0^2-4)} = {}^0/_{-4} = 0$.
   (e) Use the results of this problem to sketch a graph of f. Be sure to label all critical points and asymptotes.



3. Let $f(x) = 2x^3 - 12x^2 - 13$.
   (a) Use a sign diagram to identify the regions where f is increasing and the regions where f is decreasing.
   We need the derivative: $f'(x) = 6x^2 - 24x$. Set it to zero to find the critical points:

$$6x^2 - 24x = 0 \quad \Longrightarrow \quad 6x(x-4) = 0 \quad \Longrightarrow \quad x = 0, 4.$$

   There are no vertical asymptotes, so the sign diagram must check around the critical points only:

$f'(x)$	pos	0	neg	0	pos
x	-1	0	1	4	5

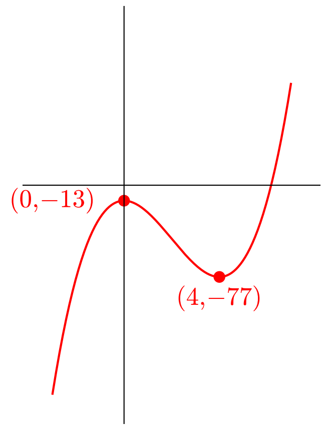
   The function is increasing on $(-\infty, 0) \cup (4, \infty)$ and decreasing on $(0, 4)$.
   (b) Use a sign diagram to identify the regions where f is concave up and the regions where f is concave down.
   We need the second derivative: $f''(x) = 12x - 24$. Set it to zero to find the possible inflection points:

$$12x - 24 = 0 \quad \Longrightarrow \quad x = 2.$$

   The sign diagram must check around the potential inflection point only:

$f''(x)$	neg	0	pos
x	0	2	3

   The function is concave up on $(2, \infty)$ and concave down on $(-\infty, 2)$.
   (c) Identify the relative maxima and relative minima.
   The sign diagram tells us that there is a relative maximum *at* $x = 0$ and a relative minimum *at* $x = 4$. Substituting this into the original function, we find the relative maximum *is* $f(0) = -13$ and the relative minimum *is* $f(4) = -77$.
   (d) Use the results of this problem to sketch a graph of f. Be sure to label all critical points and inflection points.



3. Suppose the average fuel economy of a particular car is $E(x) = -.01x^2 + 0.8x + 7.3$, where x is the driving speed in miles per hour ($20 \leq x \leq 60$). Use Calculus to determine the speed at which fuel economy is greatest.
We need to take the derivative and solve for the root(s). The derivative is $E'(x) = -.02x + 0.8$, and the root occurs at $x = 40$. Fuel economy is greatest at 40 miles per gallon.

4. Suppose the price function for a widget is $p(x) = -10x + 1000$, the fixed costs are \$450, and the variable costs are \$20 per unit.
   (a) Determine the revenue function.
   Revenue is the product of price and number of units sold; that is, $R(x) = x \cdot p(x)$. So $R(x) = -10x^2 + 1000x$.
   (b) Determine the profit function.
   Profit, as noted above, is $R(x) - C(x)$. We just found R, and C is the sum of variable and fixed costs. Each unit costs \$20, so variable costs are $VC(x) = 20x$. Total cost is the sum of fixed and variable cost, so $C(x) = 450 + 20x$. Hence the profit function is $P(x) = (-10x^2 + 1000x) - (450 + 20x)$, or $P(x) = -10x^2 + 980x - 450$.
   (c) Find the value of x that maximizes profit.
   We need to solve for where the derivative equals zero. The derivative is
   $$P'(x) = -20x + 980.$$
   Set this to zero and solve, to get
   $$x = {}^{980}/_{20} = 49.$$
   So the value of x that maximizes profit is 49 widgets.
   (d) Determine the maximum profit, and the price of the widget at that level.
   We found that profix it maximized when we produce 49 widgets, so the maximum profit is $P(49) = -10 \cdot 49^2 + 80 \cdot 49 - 450 = 23560$, or \$23,560. The price of the widget at that level is $p(49) = 510$, or \$510.

5. A dealer has noticed that during peak sales season he can typically sell each day 4 units of a certain model of car for \$33,000. A market research agency tells him that for every \$3,000 price cut, he can sell another two cars, as well. The manufacturer charges him \$24,000 per car, while property taxes, utilities, and salaried employees have him on the hook for about \$2,000

every day, anyway. What price should he set in order to maximize profits on the car, and what will that maximum profit be?

Let x represent the number of cars sold. This problem does not give us a price function, so the first thing we need to do is find the price function. The information we do have is a point, $(4, 33000)$, and a slope, $-3000/2 = -1500$, so the function must be linear. We use the point-slope form of the line to find the equation,

$$y - 33000 = -1500(x-4)$$
$$y = -1500(x-4) + 33000,$$

so the price function must be

$$p(x) = -1500x + 39000.$$

That gives us a revenue function of

$$R(x) = x \cdot p(x) = -1500x^2 + 39000x.$$

We also need the cost equation. Variable costs are the cost per car; that is, $VC(x) = 24000x$. Fixed costs are the \$2000 he has to pay, anyway, so the total cost is

$$C(x) = 24000x + 2000.$$

This allows us to identify the profit function,

$$P(x) = R(x) - C(x) = \left(-1500x^2 + 39000x\right) - (24000x + 2000) = -1500x^2 + 15000x - 2000.$$

To maximize profit, we look for critical points,

$$0 = P'(x) = -3000x + 15000$$
$$x = 5.$$

So the dealer maximizes profit when he sells 5 cars. The price he must set is $p(5) = 31500$, and the profit he reaps is $P(5) = 35500$.