The skeletons you find when you order your ideal's closet

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Overview

Ordering your ideal's closet

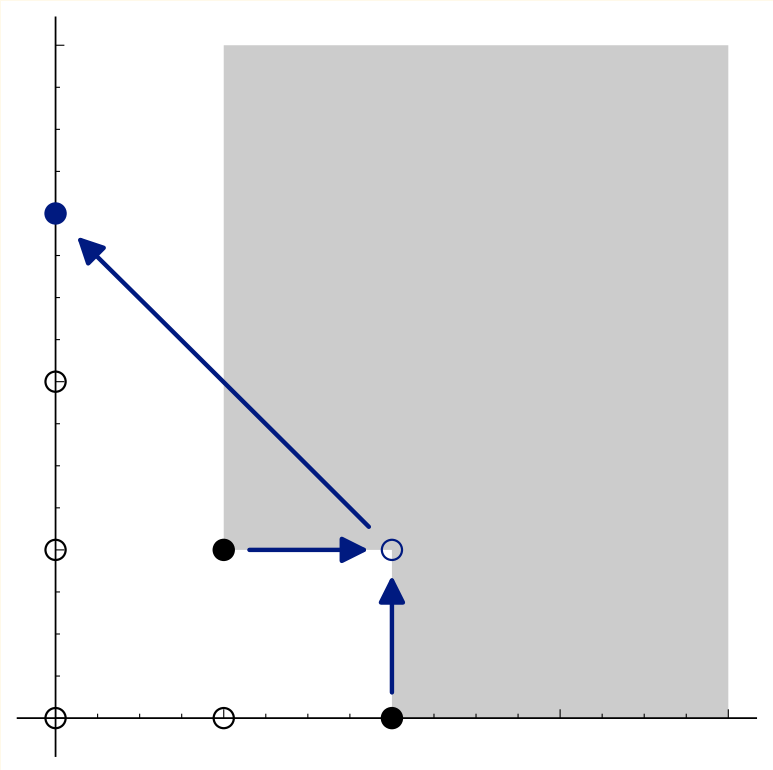
Sorting the closet

Skeletons in the closet

Conclusion

§1. Ordering your ideal's closet

New elements come from ideal's "closet"

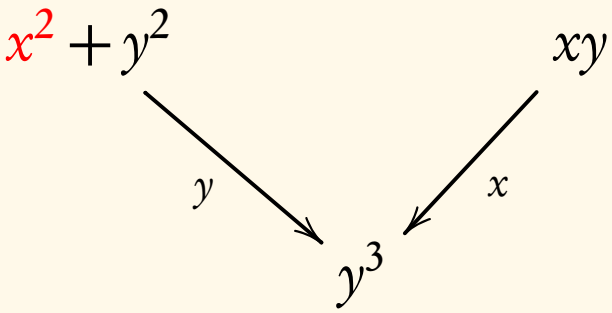


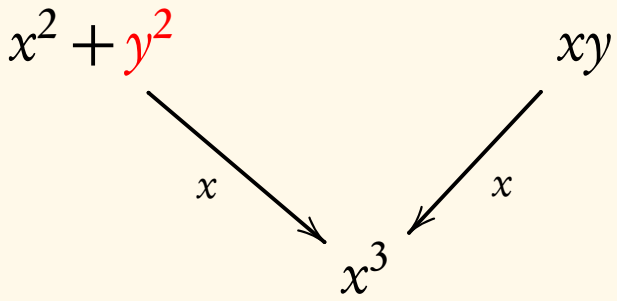
(Ideal of leading terms)

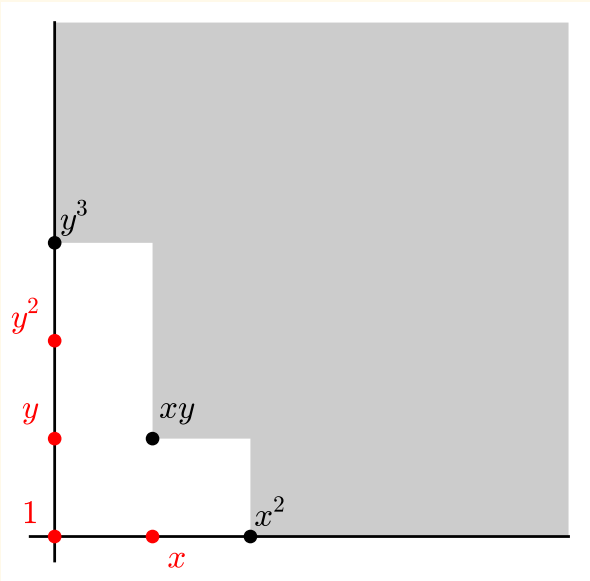
Usefulness

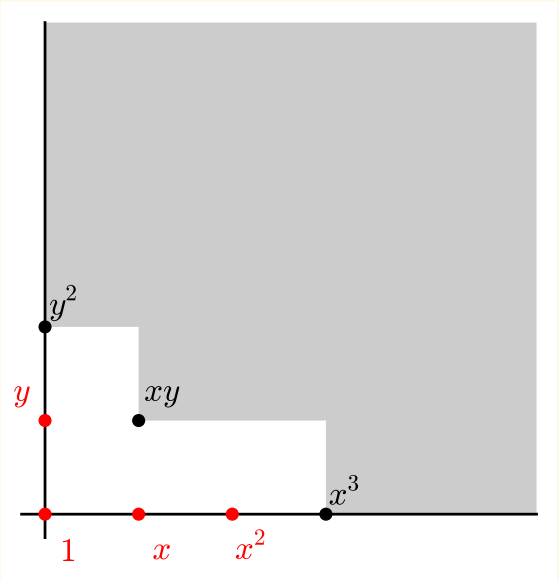
- algebraic geometry
- analysis of codes
- astrophysics
- commutative algebra
- computing discrete logarithms
- cryptanalysis
- learning with errors
- …

Different orders









Some orderings "better"

Cyclic-5

$$\left\{ \begin{array}{c} x_1 + x_2 + \cdots + x_5, \\ x_1x_2 + x_2x_3 + \cdots + x_5x_1, \\ x_1x_2x_3 + x_2x_3x_4 + \cdots + x_5x_1x_2, \\ x_1x_2x_3x_4 + x_2x_3x_4x_5 + \cdots + x_5x_1x_2x_3, \\ x_1x_2x_3x_4x_5 - 1 \end{array} \right\}$$

lex order: 11
grevlex order: 20

Another example

Cyclic-7 homogeneous

$$\left\{ \begin{array}{c} x_1 + x_2 + \cdots + x_7, \\ x_1x_2 + x_2x_3 + \cdots + x_1x_1, \\ x_1x_2x_3 + x_2x_3x_4 + \cdots + x_7x_1x_2, \\ \vdots \\ x_1x_2x_3x_4x_5x_6x_7 - h^7 \end{array} \right\}$$

lex order: 985
grevlex order: 443

Another example

Cyclic-7 homogeneous

$$\left\{ \begin{array}{c} x_1 + x_2 + \cdots + x_7, \\ x_1x_2 + x_2x_3 + \cdots + x_1x_1, \\ x_1x_2x_3 + x_2x_3x_4 + \cdots + x_7x_1x_2, \\ \vdots \\ x_1x_2x_3x_4x_5x_6x_7 - h^7 \end{array} \right\}$$

lex order: 985
grevlex order: 443
mystery order: 118

Question

Find ~~*best*~~ "good" ordering *while* sorting?

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Dynamic Gröbner basis computation

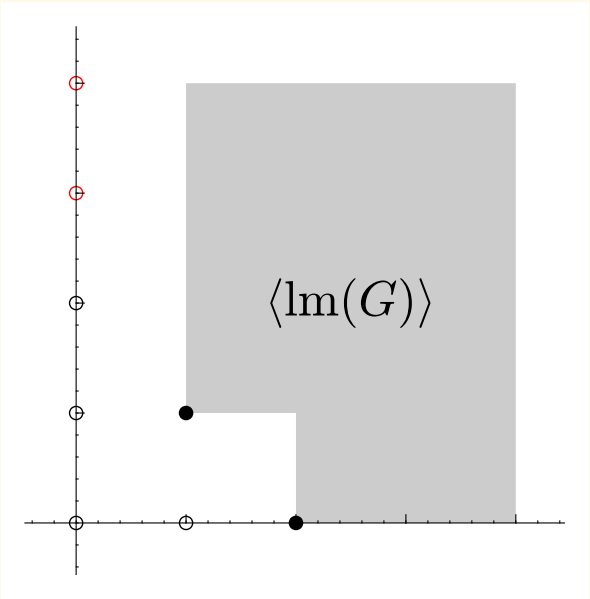
("sorting ideal's closet")
(reorder Macaulay matrix during reduction)

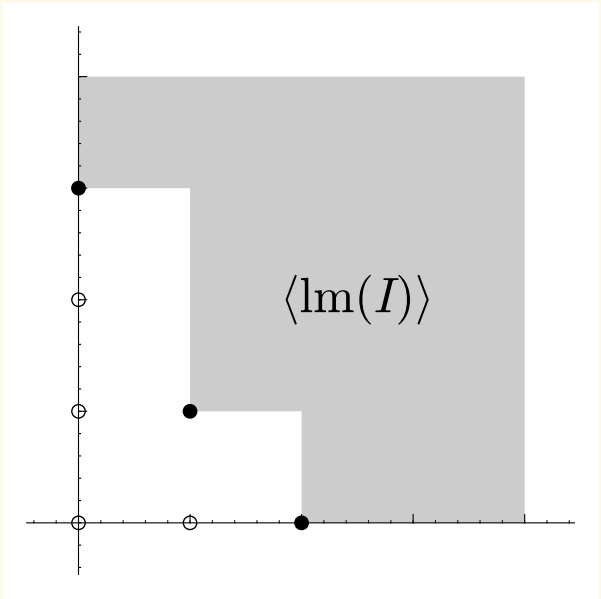
Find ~~*best*~~ "good" ordering *while* sorting?

Dynamic Gröbner basis computation

("sorting ideal's closet")
(reorder Macaulay matrix during reduction)

"Tentative" Hilbert function selects ordering





Usefulness

- smaller basis $\Longrightarrow$ easier to work with
- less computation $\Longrightarrow$ quicker result
- applied mathematicians $\Longrightarrow$ much happier

“Short” example

$$F = \left\{ \begin{array}{c} x^3y^2 + 5776x^2y^3 + 1230x^4 + 4073x^2y^2 - 1332xy^3, \\ \\ x^4y + 3199x^3 + 11994xy^2 - 8996y^2 - 10725 \end{array} \right\}$$

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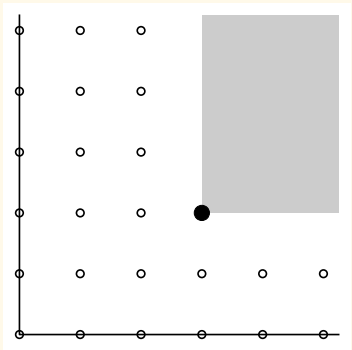
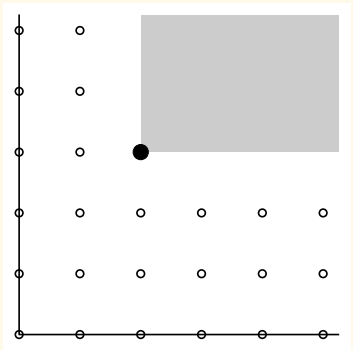
$$F = \left\{ \begin{array}{c} x^3y^2 + 5776x^2y^3 + 1230x^4 + 4073x^2y^2 - 1332xy^3, \\ \\ x^4y + 3199x^3 + 11994xy^2 - 8996y^2 - 10725 \end{array} \right\}$$

for f_1,

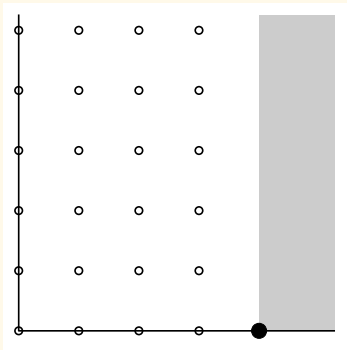
- only x^3y^2, x^2y^3, x^4 possible
- x^4 more attractive...



1,2,3,4,5,...



1,2,3,4,5,...



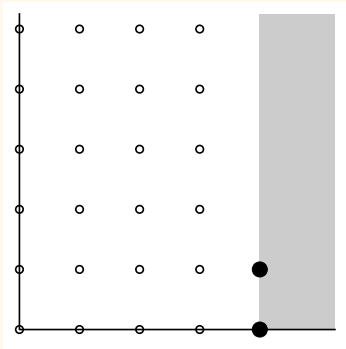
1,2,3,4,4,...

"Short" example

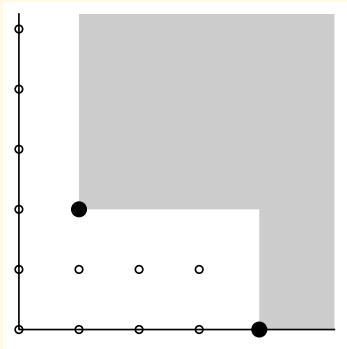
$$F = \left\{ \begin{array}{c} x^3y^2 + 5776x^2y^3 + 1230x^4 + 4073x^2y^2 - 1332xy^3, \\ \\ x^4y + 3199x^3 + 11994xy^2 - 8996y^2 - 10725 \end{array} \right\}$$

for f_2,

- only x^4y, xy^2 possible
- xy^2 more attractive...



1,2,3,4,4,...



1,2,3,3,2,1,1,...

...but *incompatible* w/choice of x^4

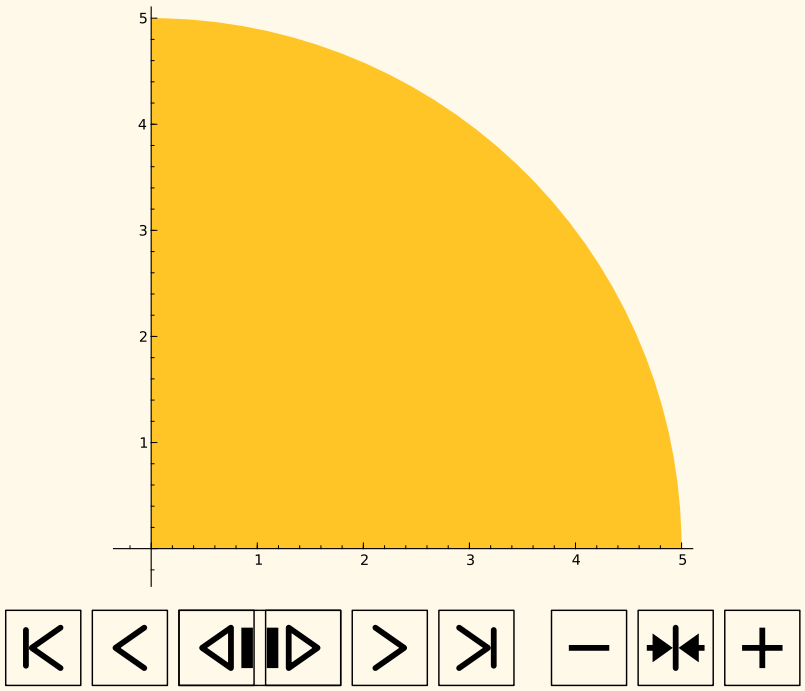
§2. Sorting the closet

Previous work

- Mora and Robbiano (1988)
  . . . pose question
- Gritzmann and Sturmfels (1993)
  . . . describe *general* algorithm w/convex geometry
- Caboara (1993)
  . . . describes, implements *refining* algorithm w/simplex
- Oleg Golubitsky (preprint)
  . . . describes, implements (?) *general* algorithm

. . . nothing else!

Refining v. general



	Refining	**General**
ordering	partial	partial or total
new polys	refine ordering	re-evaluate
leading mons	preserved	can change

Refinement: pros, cons

Cons

- sometimes forego best orderings

Pros

- no need to re-evaluate previous polynomials
- no need to re-evaluate *S*-polynomials
- can still predict zero reductions
- warm-start simplex algorithm, not cold-start

Algebra of orderings

ordering

$\downarrow$

vector (ω)

Algebra of orderings

ordering

$\downarrow$

vector (ω)

$$x^2 > y^2 \qquad \Longleftrightarrow \qquad \omega \cdot (2,0) > \omega \cdot (0,2)$$

Algebra of orderings

ordering

$\downarrow$

vector (ω)

$$x^2 > y^2 \qquad \Longleftrightarrow \qquad \omega \cdot (2,0) > \omega \cdot (0,2)$$

$$\omega = (1,0) \quad x^3 + x^2y^2 + y^3$$

$$\omega = (1,1) \quad x^3 + x^2y^2 + y^3$$

$$\omega = (0,1) \quad x^3 + x^2y^2 + y^3$$

Clutter in the closet

$$x^5 + x^4y^2 + wz^3 + w^3z + wx^4 + w^2yz^2 + x^2$$

new leading polynomial	$\longrightarrow$	new constraints
many/large polynomials	$\longrightarrow$	many constraints
more constraints	$\longrightarrow$	more computation
more computation	$\longrightarrow$	unhappy applied mathematicians

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Big-time question

How can we reduce the number of constraints?

Many closet elements “strung together”

Some elements drag others along

$$x^5 \text{———} x^2$$

Divisibility criterion

$t \mid u$? no need to consider t.

(Caboara, 1993)

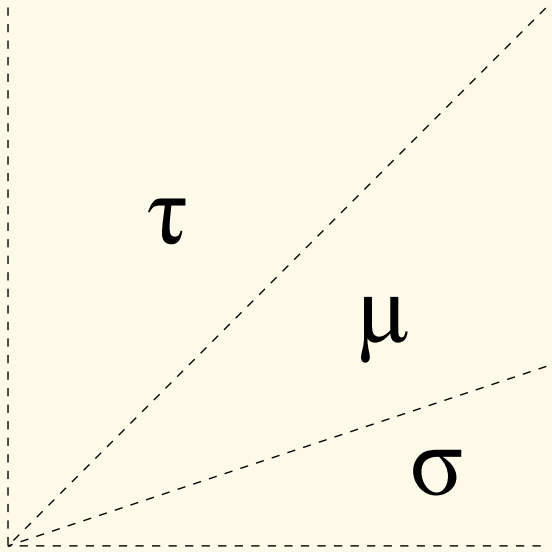
Limited usefulness

- homogeneous polynomials
- not-so-homogeneous polynomials

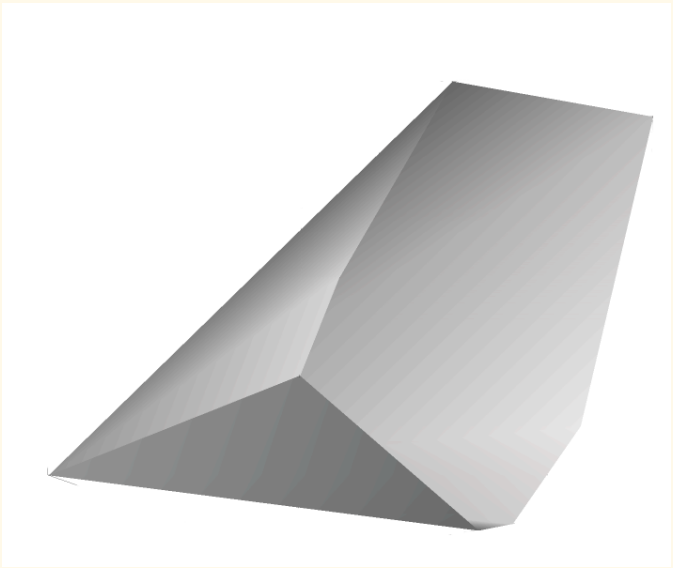
§3. Skeletons in the closet

Geometric view

linear inequalities $\longrightarrow$ polyhedral cone



2d



3d

Definition

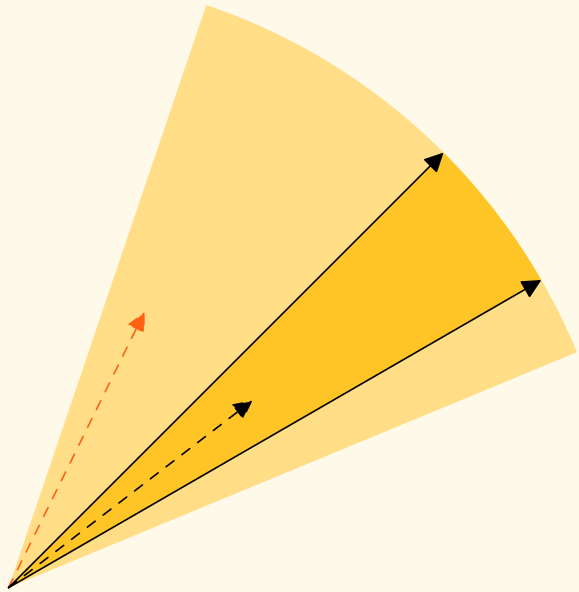
Skeleton: extreme vectors & adjacency relationship

Usefulness

Fact

Skeleton identifies minimal set of closet elements

("Easily" proved from Corner Point Theorem)



Example

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Select $x^4 = \mathrm{lm}(f_1)$

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- Skeleton: $\{(1,0),(2,1)\}$
  - let $\sigma = (1,0)+(2,1) = (3,1)$

Example

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- Skeleton: $\{(1,0),(2,1)\}$
  - let $\sigma = (1,0) + (2,1) = (3,1)$
- $f_2 \xrightarrow[f_1]{} 2784x^3y^3 + 14878x^2y^4 + 10170x^2y^3 - 13675xy^4$
  $+ 3199x^3 + 11994xy^2 - 899y^2 - 10725$

Example

$$F = \left\{ \begin{array}{c} x^3y^2 + 5776x^2y^3 + 1230x^4 + 4073x^2y^2 - 1332xy^3, \\ \\ x^4y + 3199x^3 + 11994xy^2 - 8996y^2 - 10725 \end{array} \right\}$$

Select $x^4 = \mathrm{lm}(f_1)$

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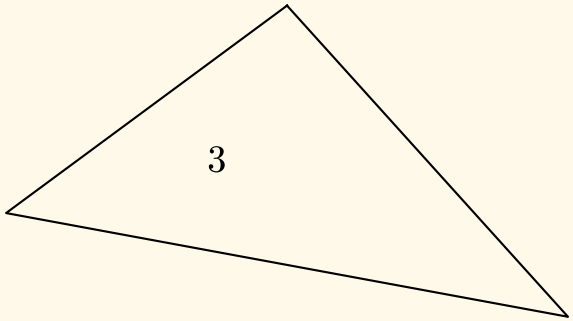
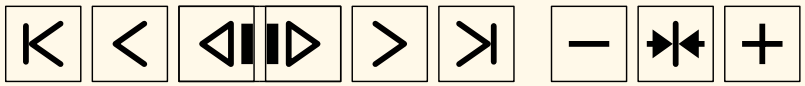
- $\mathrm{lm}_\sigma(f_2) = x^3y^3$
  - $t \not> _\tau \mathrm{lm}_\sigma(f_2)$ for $t \in f_2$, $\tau \in \{(1,0),(2,1)\}$

  $\therefore$ no other possible lms

Should we use the *entire* skeleton?

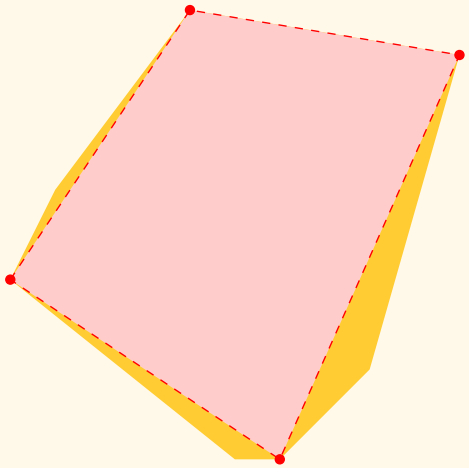




- Explosion in # of vectors?
- "Hard" to compute? (rel. simplex)

“Approximate” skeleton?

compute only max, min ω_i



disallow some needed, allow only needed

- $O(n)$ space
- $O(n)$ “easy” calls to simplex: change objective function

Practical results

	#constraints		size of basis (pols × mons)	
	div only	app skel	dyn	grevlex
Caboara 1	29	20	35 × 137	239 × 478
Caboara 2	17	26	21 × 42	553 × 1106
Cyclic-5	61	31	11 × 68	20 × 85
Cyclic-6	250	56	20 × 129	45 × 199
Cyclic-7	>1000	63	63 × 1049	209 × 1134
Cyclic-5 hom	60	33	11 × 98	38 × 197
Cyclic-6 hom	303	39	33 × 476	99 × 580
Cyclic-7 hom	>1000	105	222 × 5181	443 × 3395

- Able to compute dyn GB we could not do before!
- Reduction consumes most time (usually)
- Katsura-n: similar size basis as static, fewer constraints

(Caboara & Perry, 2014)

You knew there was a penalty somewhere

CAS's require *integer* orderings

- simplex *fast* when *inexact*
- *integer* solutions ***slower than molasses in winter***
    - narrow cone? large integer entries

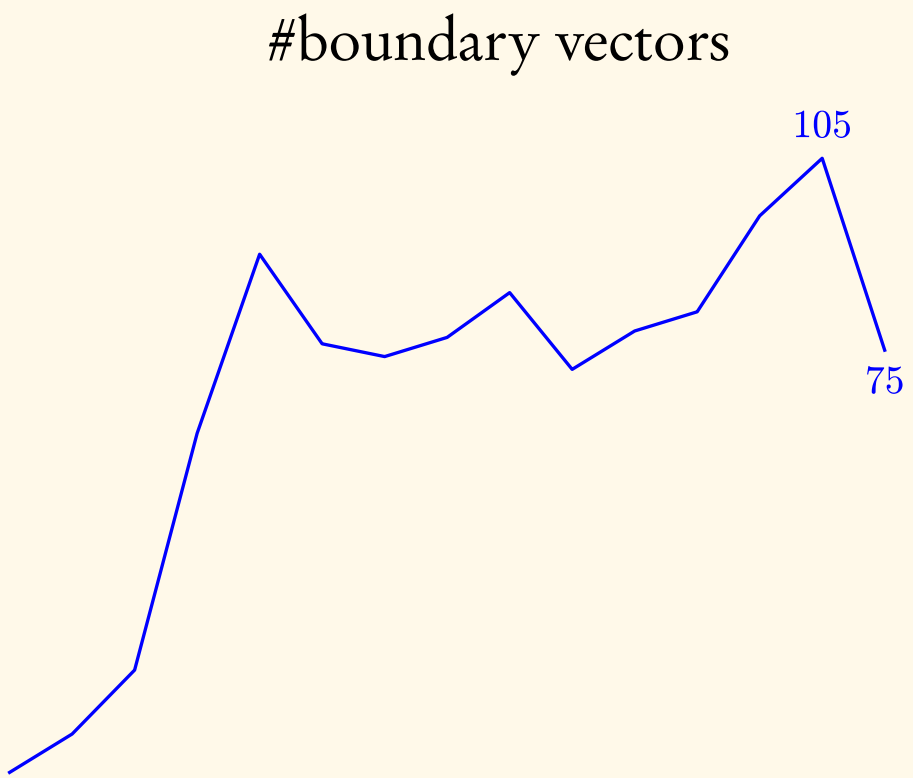
(CAS also don't like *large* integer products)

Life is short. Why not?

Double description method

- Exact
- Iterative: well-suited for refinement!
  - add constraints as needed
  - detects, discards redundant constraints
- Well-studied
  - Motzkin et al., 1953
  - Fukuda and Prodon, 1996
  - Zolotych, 2012

Evolution of skeleton (1)



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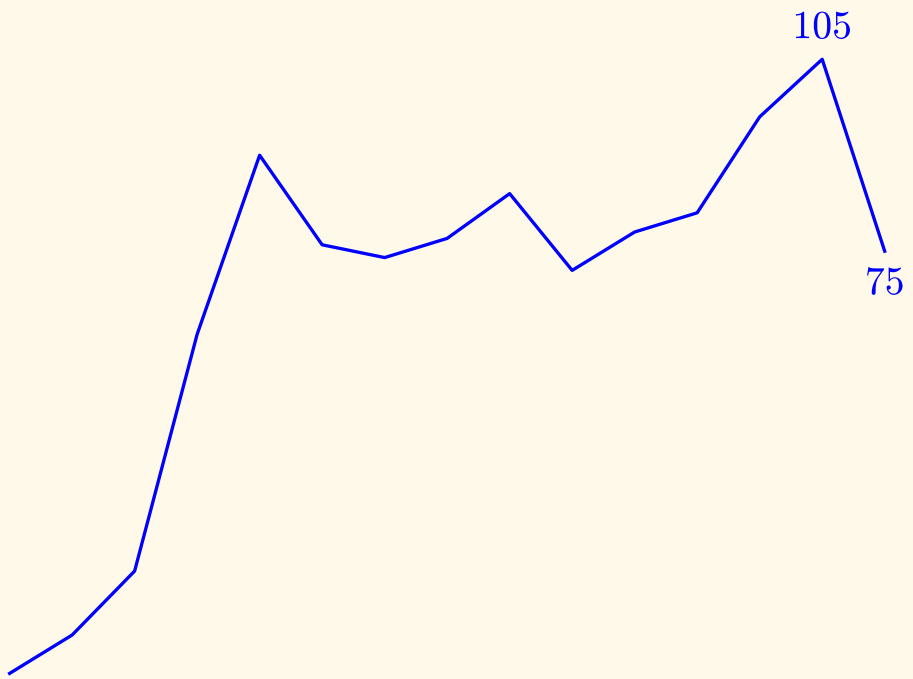
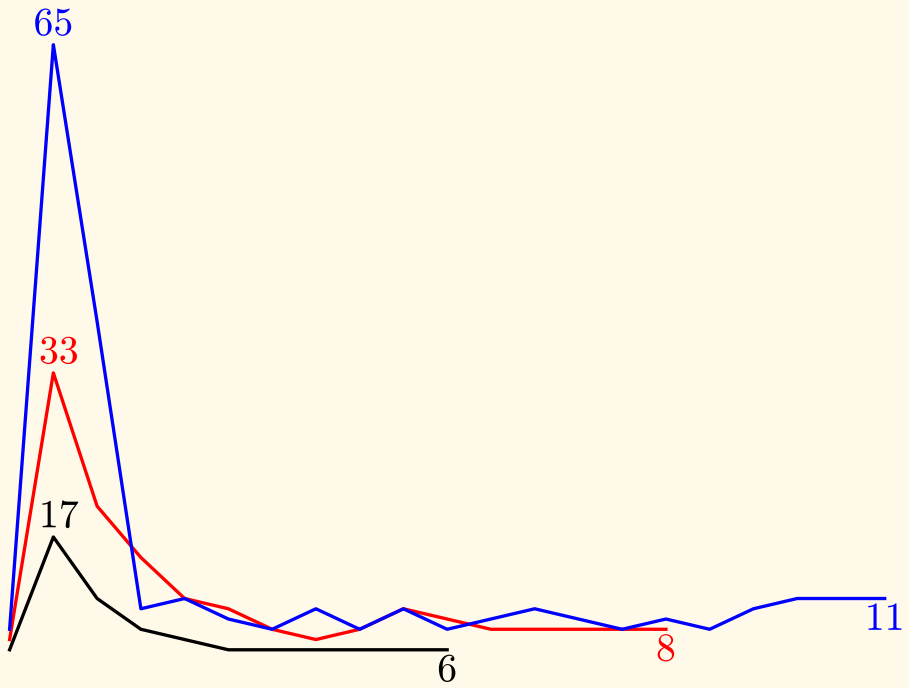


bad news:

- finding leading monomials $\gg$ reduction
- larger GB than using approximate skeleton
  - greedy algorithm does not pay here
  - still smaller than static (grevlex)
- similar for Caboara 2

Evolution of skeleton (2)

#boundary vectors



Cyclic-5 Cyclic-6 Cyclic-7

(all homogeneous)

Yes, entire skeleton!

Cylic-7 homogeneous

	grevlex	app skel	ent skel
#polys	443	222	126*
#mons	3395	5181	4312
#vectors	N/A	4	11
#constraints	N/A	105	74

*118 non-redundant

Using entire skeleton:

- *much* smaller basis
- costliest stage? reduction!
  - *not* selecting ordering!
  - *no integer programming*
- not many boundary vectors!

§4. Conclusion

The perfect is the enemy of the good

- We obtain *good* orderings from dynamic algorithm
  - not necessarily *best* ordering
  - what makes "best"? shortest? fastest?
- Approximate v. exact skeleton

Challenges

Data structures

- Integer orderings grow large
- CAS require integer orderings
- Implemented as "marked" polynomials

Reduction

- not all CAS use Sugar

Choosing good heuristic

Future

- Efficient implementation
  - Hilbert polynomial & series
  - Double description method
  - Compiled (*not Cython*, or *better Cython*)
- Better (?) criteria to select ordering
- Combine w/modern techniques

Thank you!

Massimo Caboara

Anonymous referee

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University of Southern Mississippi

Галина Валеревна Заморий

Further reading

- Mora and Robbiano, "The Gröbner fan of an ideal." *JSC* 6 (1988) 183–208.
- Grtizmann and Sturmfels, "Minkowski Addition of Polytopes: Computational Complexity and Applications to Gröbner Bases." *SIAM J. Disc. Math* 6 (1993) 246–269.
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- O. Golubitsky, "Converging term order sequences and the dynamic Buchberger algorithm." In preparation, preprint received via private communication.
- Caboara and Perry, "Reducing the size and number of linear programs in a dynamic Gröbner basis algorithm." *AAECC* 25 (2014) 99–117.