Combinatorial Criteria for Groebner Bases

or, how to solve the world's problems, one step at a time

Jack Perry

PhD Student (Computer Algebra)

NC State University

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Outline (1)

- Overview of Groebner Bases
- Definition of “Combinatorial Criteria”
- Why we care
- Buchberger’s First and Second Criteria
- Hong’s Criterion for Groebner bases under Composition

Outline (2)

- Improving Hong's Criterion: Other Known Criteria
- Improving Hong's Criterion: New Result and Conjectured Criterion
- "Improving" Hong's Criterion: New Result
- Improving Buchberger's Second Criterion: Conjectured Criterion

Step One:

An Introduction to Groebner Bases

Groebner Bases (1)

1. How do I solve $x + b = 0$?

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3. How do I tell a computer to do this?

1. How do I solve this system?

$$
\begin{aligned}
x + b_1 y + c_1 &= 0 \\
x + b_2 y + c_2 &= 0
\end{aligned}
$$

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$$\begin{aligned} x + b_1 y + c_1 &= 0 \\ x + b_2 y + c_2 &= 0 \end{aligned}$$

2. How do I extend it to this one?

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3. How do I tell a computer to do this?

Example:

Solve

$$\begin{aligned} 105x - 6y &= -21 \\ -70x + 4y &= 14 \end{aligned}$$

Method:

$$\frac{\mathrm{lcm}\left(\mathrm{lt}_x(1), \mathrm{lt}_x(2)\right)}{\mathrm{lt}_x(1)}(1) - \frac{\mathrm{lcm}\left(\mathrm{lt}_x(1), \mathrm{lt}_x(2)\right)}{\mathrm{lt}_x(2)}(2)$$

Example:

Solve

$$\begin{aligned} 105x - 6y &= -21 \quad (1) \\ -70x + 4y &= 14 \quad (2) \end{aligned}$$

Method:

$$\frac{\mathsf{lcm}\left(\mathsf{lt}_x(1), \mathsf{lt}_x(2)\right)}{\mathsf{lt}_x(1)}(1) - \frac{\mathsf{lcm}\left(\mathsf{lt}_x(1), \mathsf{lt}_x(2)\right)}{\mathsf{lt}_x(2)}(2)$$

reduces system to

$$ay = b$$

Example:

Obtain

$$
\begin{aligned}
(-210x + 12y &= 42) \\
-\,(-210x + 12y &= 42) \\
&\Downarrow \\
0y &= 0
\end{aligned}
$$

$\therefore$ ***Infinitely many solutions.***

Observe:

Solving linear system

$\Downarrow$

Rewriting as "nicer",

"equivalent" system

Observe:

“Nicer” form yields

(in degree 1)

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"Nicer" form yields

1. Dimension of solution space

(in degree 1)

Observe:

"Nicer" form yields

1. Dimension of solution space
2. Quicker Solving

(in degree 1)

How do I solve this system?

$$
\begin{aligned}
x^2 + b_1 x f_1(y) + c_1 g_1(y) &= 0 \\
x^2 + b_2 x f_2(y) + c_2 g_2(y) &= 0
\end{aligned}
$$

How do I solve this system?

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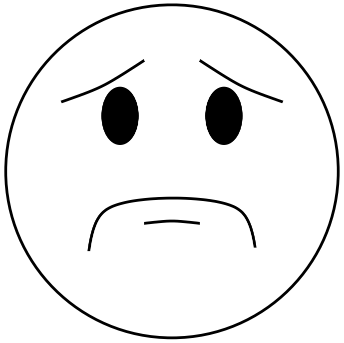
$$



Tool:

Groebner Bases

$$\Downarrow$$

Tool:

Groebner Bases

$\Downarrow$

"nicer", "equivalent" system

for polynomial ideals

Precise Definition:

G **is a Groebner Basis**

$$\Updownarrow$$

$$\forall i \neq j \quad S_{ij} \;:=\; \frac{\mathrm{lcm}\left(\mathrm{lt}\left(g_i\right), \mathrm{lt}\left(g_j\right)\right)}{\mathrm{lt}\left(g_i\right)} g_i - \frac{\mathrm{lcm}\left(\mathrm{lt}\left(g_i\right), \mathrm{lt}\left(g_j\right)\right)}{\mathrm{lt}\left(g_j\right)} g_j$$

$$= \sum_{k=1}^{r} h_k g_k$$

$$\forall k \; h_k = 0 \text{ or}$$

$$\mathrm{lt}\left(h_k\right) \mathrm{lt}\left(g_k\right) < \mathrm{lcm}\left(\mathrm{lt}\left(g_i\right), \mathrm{lt}\left(g_j\right)\right)$$

Notation:

The current notation is too unwieldy. So we write

$$\overline{g_i} \;\leftrightarrow\; \mathsf{lt}\,(g_i)$$

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The current notation is too unwieldy. So we write

$$
\begin{aligned}
\overline{g_i} &\Leftrightarrow \mathsf{lt}\left(g_i\right) \\
[p, q] &\Leftrightarrow \mathsf{lcm}\left(p, q\right) \\
\sigma_{ij} &\Leftrightarrow \frac{\left[\overline{g_i}, \overline{g_j}\right]}{\overline{g_i}}
\end{aligned}
$$

Notation:

The current notation is too unwieldy. So we write

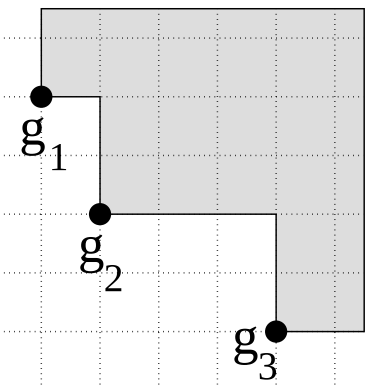
$$S_{ij} \to 0 \;\Leftrightarrow\; \exists h_k \quad S_{ij} = \sum h_k g_k$$
$$\wedge \forall k \left[h_k = 0 \quad \vee \quad \overline{h_k}\,\overline{g_k} < \left[\overline{g_i}, \overline{g_j}\right]\right]$$

Precise Definition:

GB (G) iff

$$\forall i \neq j \quad \left(S_{ij} := \sigma_{ij} g_i - \sigma_{ji} g_j\right) \to 0.$$

Picture Definition:



No combinations of g_i in white zone!

Example:

$$G = \left\{ x^2 y + 1, xy^2 + 1 \right\}$$

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But

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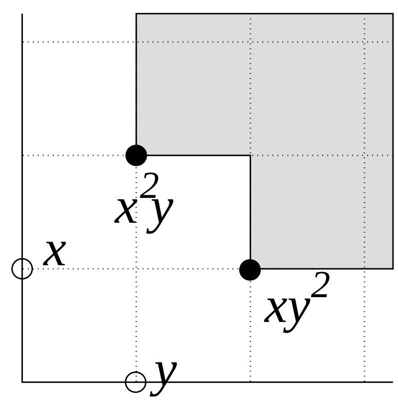
But

$$\neg\exists h_1, h_2 \in k\,[x, y] \qquad -x + y = h_1 g_1 + h_2 g_2$$

and

$$h_k \neq 0 \Rightarrow \overline{h_k}\,\overline{g_k} < [\overline{g_1}, \overline{g_2}]\,.$$

Example:



Interlude:

Term Orderings

Difference from univariate:

$\overline{g_i}$ not necessarily ***unique***

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Example:

$$\overline{x + y^3 + 1} = \mathbf{?}$$

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Example:

$$\overline{x + y^3 + 1} = x\text{?}$$

Difference from univariate:

$\overline{g_i}$ not necessarily ***unique***

Example:

$$\overline{x + y^3 + 1} = y^3\textbf{?}$$

Difference from univariate:

$\overline{g_i}$ not necessarily ***unique***

Example:

$$\overline{x + y^3 + 1} = 1\textbf{?}$$

Definition:

$>$ is an ***admissible term ordering*** iff
$\forall$ terms p, q, r:

1.

2.

3.

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1. $p \neq q \Rightarrow p > q$ or $p < q$
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Definition:

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$\forall$ terms p, q, r:

1. $p \neq q \Rightarrow p > q$ or $p < q$
2. $p|q \Rightarrow p \leq q$
3. $p > q \Rightarrow pr > qr$

Consequence:

$$\forall t \; 1 | t$$

$$\Downarrow$$

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$$\forall t \; 1 | t$$

$$\Downarrow$$

$$[\overline{f} = 1 \;\Rightarrow\; f \text{ constant}]$$

Fact:

$$\forall >$$

$$\exists M \in \mathbb{R}^{r \times r} \quad |M| \neq 0$$

“representing” $>$

Example 1:

pure lexicographic ordering

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$$\Rightarrow \quad x > y^3$$

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$$\Rightarrow \quad M = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Example 2:

total-degree ordering

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$$\Rightarrow \quad x < y^3$$

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Drawback:

Solving systems? $\longrightarrow$ lex

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Answer sometime this century? $\longrightarrow$ tdeg

One Solution: “Groebner Walk”

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1. Compute GB $\left(G, >_{\mathrm{tdeg}}\right)$

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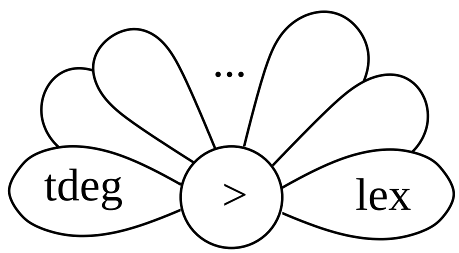
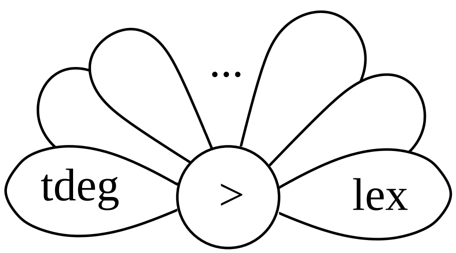
2. Groebner Fan: 

One Solution: "Groebner Walk"

1. Compute GB $\left(G, >_{\text{tdeg}}\right)$

2. Groebner Fan:



3. "Walk": $>_{\text{tdeg}} \longrightarrow >_{\text{lex}}$

“Combinatorial Criteria”?

Real World

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Real World

$\Downarrow$

Structured Problems

“Combinatorial Criteria”?

Real World

$\Downarrow$

Structured Problems

$\Downarrow$

Ignore pathological cases

“Combinatorial Criteria”?

Real World

$\Downarrow$

Structured Problems

$\Downarrow$

Ignore pathological cases

$\Downarrow$

Exponent structure

Why we care

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1. PhD thesis, career, big bucks, &c.

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   time intensive

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   (blame polynomial division)

Why we care

1. PhD thesis, career, big bucks, &c.
2. Computing GB (G):

   time intensive

   memory intensive

   (blame polynomial division)
3. Combinatorial criteria $\Rightarrow$ ***FAST!!!***

Step Two:

The First Results

Buchberger's First Criterion

$$\forall i \neq j \qquad \left(\overline{g_i}, \overline{g_j}\right) = 1$$

$$\Downarrow$$

Buchberger's First Criterion

$$\forall i \neq j \qquad \left(\overline{g_i}, \overline{g_j}\right) = 1$$

$$\Downarrow$$

$$S_{ij} \rightarrow 0$$

Example:

$$G = \left\{x^5 - 4x^2w + 4v, wy^{10} - wy, z^5 - z^2v\right\}$$

GB (G) for both lex ($x > y > z > w > v$) and tdeg.

Why?

$$g_i = \overline{g_i} + R_i \qquad g_j = \overline{g_j} + R_j$$

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$$\left(\overline{g_i}, \overline{g_j}\right) = 1 \quad \Rightarrow \quad S_{ij} = \overline{g_j} g_i - \overline{g_i} g_j$$

Why?

$$g_i = \overline{g_i} + R_i \qquad g_j = \overline{g_j} + R_j$$

$$\left(\overline{g_i}, \overline{g_j}\right) = 1 \quad \Rightarrow \quad S_{ij} = \overline{g_j} g_i - \overline{g_i} g_j$$

$$\begin{aligned} \therefore S_{ij} &= \overline{g_j} R_i - \overline{g_i} R_j \\ &= \overline{g_j} R_i + R_j R_i - \overline{g_i} R_j - R_j R_i \end{aligned}$$

Why?

$$g_i = \overline{g_i} + R_i \qquad g_j = \overline{g_j} + R_j$$

$$\left(\overline{g_i}, \overline{g_j}\right) = 1 \quad \Rightarrow \quad S_{ij} = \overline{g_j} g_i - \overline{g_i} g_j$$

$$\begin{aligned} \therefore S_{ij} &= \overline{g_j} R_i + R_j R_i - \overline{g_i} R_j - R_i R_j \\ &= R_i g_j - R_j g_i \end{aligned}$$

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$$g_i = \overline{g_i} + R_i \qquad g_j = \overline{g_j} + R_j$$

$$\left(\overline{g_i}, \overline{g_j}\right) = 1 \quad \Rightarrow \quad S_{ij} = \overline{g_j} g_i - \overline{g_i} g_j$$

$$\begin{aligned} \therefore S_{ij} &= \overline{g_j} R_i + R_j R_i - \overline{g_i} R_j - R_i R_j \\ &= R_i g_j - R_j g_i \end{aligned}$$

$$!!! \quad \overline{R_j}, \overline{R_i} < \left[\overline{g_i}, \overline{g_j}\right]$$

Buchberger's Second Criterion

$$\overline{g_k} \,\Big|\, \left[\overline{g_i}, \overline{g_j}\right]$$

$$\Downarrow$$

Buchberger's Second Criterion

$$\overline{g_k} \mid \left[\overline{g_i}, \overline{g_j}\right]$$

$$\Downarrow$$

$$S_{ik} \longrightarrow 0 \wedge S_{kj} \longrightarrow 0 \Rightarrow S_{ij} \longrightarrow 0$$

Example:

$$G = \left\{x^2 + R_1, y^2 + R_2, xy + R_3\right\}$$

Example:

$$G = \left\{ x^2y + R_1, xy^2 + R_2, xy + R_3 \right\}$$

$S_{13} \longrightarrow 0$ and $S_{23} \longrightarrow 0$

$$\Downarrow$$

$S_{12} \longrightarrow 0$ for *free!!!*

Why?

$$S_{ij} = \frac{\left[\overline{g_i}, \overline{g_j}\right]}{\overline{g_i}} g_i - \frac{\left[\overline{g_i}, \overline{g_j}\right]}{\overline{g_j}} g_j$$

Why?

$$
\begin{aligned}
S_{ij} &= \frac{\left[\overline{g_i}, \overline{g_j}\right]}{\overline{g_i}} g_i - \frac{\left[\overline{g_i}, \overline{g_j}\right]}{\overline{g_j}} g_j \\
&= \frac{\left[\overline{g_i}, \overline{g_j}\right]}{[\overline{g_i}, \overline{g_k}]} \cdot \frac{[\overline{g_i}, \overline{g_k}]}{\overline{g_i}} g_i - \frac{\left[\overline{g_i}, \overline{g_j}\right]}{\left[\overline{g_j}, \overline{g_k}\right]} \cdot \frac{\left[\overline{g_j}, \overline{g_k}\right]}{\overline{g_j}} g_j
\end{aligned}
$$

Why?

$$
\begin{aligned}
S_{ij} &= \frac{\left[\overline{g_i},\overline{g_j}\right]}{\left[\overline{g_i},\overline{g_k}\right]}\cdot\frac{\left[\overline{g_i},\overline{g_k}\right]}{\overline{g_i}}g_i-\frac{\left[\overline{g_i},\overline{g_j}\right]}{\left[\overline{g_j},\overline{g_k}\right]}\cdot\frac{\left[\overline{g_j},\overline{g_k}\right]}{\overline{g_j}}g_j\\
&= \frac{\left[\overline{g_i},\overline{g_j}\right]}{\left[\overline{g_i},\overline{g_k}\right]}\cdot\frac{\left[\overline{g_i},\overline{g_k}\right]}{\overline{g_i}}g_i-\frac{\left[\overline{g_i},\overline{g_j}\right]}{\left[\overline{g_i},\overline{g_k}\right]}\cdot\frac{\left[\overline{g_i},\overline{g_k}\right]}{\overline{g_k}}g_k\\
&\quad+\frac{\left[\overline{g_i},\overline{g_j}\right]}{\left[\overline{g_j},\overline{g_k}\right]}\cdot\frac{\left[\overline{g_j},\overline{g_k}\right]}{\overline{g_k}}g_k-\frac{\left[\overline{g_i},\overline{g_j}\right]}{\left[\overline{g_j},\overline{g_k}\right]}\cdot\frac{\left[\overline{g_j},\overline{g_k}\right]}{\overline{g_j}}g_j
\end{aligned}
$$

Why?

$$\begin{aligned}
S_{ij} &= \frac{\left[\overline{g_i}, \overline{g_j}\right]}{[\overline{g_i}, \overline{g_k}]} \cdot \frac{[\overline{g_i}, \overline{g_k}]}{\overline{g_i}} g_i - \frac{\left[\overline{g_i}, \overline{g_j}\right]}{[\overline{g_i}, \overline{g_k}]} \cdot \frac{[\overline{g_i}, \overline{g_k}]}{\overline{g_k}} g_k \\
&\quad + \frac{\left[\overline{g_i}, \overline{g_j}\right]}{\left[\overline{g_j}, \overline{g_k}\right]} \cdot \frac{\left[\overline{g_j}, \overline{g_k}\right]}{\overline{g_k}} g_k - \frac{\left[\overline{g_i}, \overline{g_j}\right]}{\left[\overline{g_j}, \overline{g_k}\right]} \cdot \frac{\left[\overline{g_j}, \overline{g_k}\right]}{\overline{g_j}} g_j \\
&= \frac{\left[\overline{g_i}, \overline{g_j}\right]}{[\overline{g_i}, \overline{g_k}]} \cdot S_{ik} + \frac{\left[\overline{g_i}, \overline{g_j}\right]}{\left[\overline{g_j}, \overline{g_k}\right]} \cdot S_{kj}
\end{aligned}$$

Composition:

$$G \subset k[x_1, \ldots, x_m]$$

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$$G \subset k[x_1, \ldots, x_m]$$

$$\Theta = (\theta_1, \ldots, \theta_m) \subset k[y_1, \ldots, y_n]$$

$$m \leq n$$

Composition:

$$
\begin{aligned}
G \;&\subset\; k[x_1, \ldots, x_m] \\
\Theta = (\theta_1, \ldots, \theta_m) \;&\subset\; k[y_1, \ldots, y_n] \\
g \circ \Theta \;&=\; g(\theta_1, \ldots, \theta_m)
\end{aligned}
$$

$$m \leq n$$

Composition:

$$\mathsf{GB}\,(G) \Rightarrow \mathsf{GB}\,(G \circ \boldsymbol{\ominus})\,?$$

Hoon Hong's Criterion ($m = n$)

$$\forall G \; \mathsf{GB}(G) \Rightarrow \mathsf{GB}(G \circ \Theta)$$

$$\Updownarrow$$

Hoon Hong's Criterion ($m = n$)

$$\forall G \ \mathsf{GB}\,(G) \Rightarrow \mathsf{GB}\,(G \circ \Theta)$$

$$\Updownarrow$$

$\overline{\Theta}$ is a list of permuted powers

Example:

$$F = \left\{ x^{30}y^{15}, x^{5}y^{25} \right\}$$

$$\Theta = \left(a^{5} + ab + 1, b^{4} - 3b^{2} - 4 \right)$$

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$$F = \left\{x^{30}y^{15}, x^{5}y^{25}\right\}$$

$$\Theta = \left(a^{5} + ab + 1, b^{4} - 3b^{2} - 4\right)$$

HHC: 0.03 seconds

Current Maple:

Example:

$$F = \left\{x^{30}y^{15}, x^{5}y^{25}\right\}$$

$$\Theta = \left(a^5 + ab + 1, b^4 - 3b^2 - 4\right)$$

HHC: 0.03 seconds

Current Maple: 193.710 seconds

6457 times faster!!!

Refinements of HHC ($m < n$)

Wang Mingsheng et al.:

$$\forall G \; \mathsf{GB}\,(G) \Rightarrow \mathsf{GB}\,(G \circ \boldsymbol{\Theta})$$

$$\Updownarrow$$

Refinements of HHC ($m \leq n$)

Wang Mingsheng et al.:

$$\forall G \; \mathsf{GB}\,(G) \Rightarrow \mathsf{GB}\,(G \circ \boldsymbol{\Theta})$$

$$\Updownarrow$$

$\overline{\boldsymbol{\Theta}}$ is a list of relatively prime terms

Example:

$$F = \left\{x^{30}y^{15}, x^{5}y^{25}\right\}$$

$$\Theta = \left(a^{5}c^{3} + ab + c^{2}, b^{4}d^{3} - 3b^{2} - 4d^{5}\right)$$

WMC: 0.049 seconds

Current Maple: 9833.261 seconds $\approx$ 164 minutes

~200,680 times faster!!!

Refinements of HHC ($m \leq n$)

Hong and Perry:

$$
\begin{gathered}
\forall G, \Theta \\
\forall i \neq j \quad \left[\overline{g_i}, \overline{g_j}\right] \circ \overline{\Theta} \;=\; \left[\overline{g_i} \circ \overline{\Theta}, \overline{g_j} \circ \overline{\Theta}\right] \\
\Downarrow
\end{gathered}
$$

Refinements of HHC ($m \leq n$)

Hong and Perry:

$$
\begin{array}{c}
\forall G, \Theta \\
\forall i \neq j \quad \left[\overline{g_i}, \overline{g_j}\right] \circ \overline{\Theta} \;=\; \left[\overline{g_i} \circ \overline{\Theta}, \overline{g_j} \circ \overline{\Theta}\right] \\
\Downarrow \\
\forall G', \Theta' \qquad \overline{G} = \overline{G'} \quad \overline{\Theta} = \overline{\Theta'} \\
\mathrm{GB}\left(G'\right) \;\Rightarrow\; \mathrm{GB}\left(G' \circ \Theta'\right)
\end{array}
$$

Definition:

$$\mathsf{LCMCOMM}\left(g_i, g_j, \ominus\right)$$

$$\Updownarrow$$

$$\left[\overline{g_i}, \overline{g_j}\right] \circ \overline{\ominus} = \left[\overline{g_i} \circ \overline{\ominus}, \overline{g_j} \circ \overline{\ominus}\right]$$

Refinements of HHC ($m \leq n$)

Hong and Perry: $\forall G, \Theta$

$$\forall i \neq j \qquad \mathsf{LCMCOMM}\left(g_i, g_j, \Theta\right)$$

$$\Downarrow$$

$$\forall G', \Theta' \qquad \overline{G} = \overline{G'} \quad \overline{\Theta} = \overline{\Theta'}$$

$$\mathsf{GB}\left(G'\right) \Rightarrow \mathsf{GB}\left(G' \circ \Theta'\right)$$

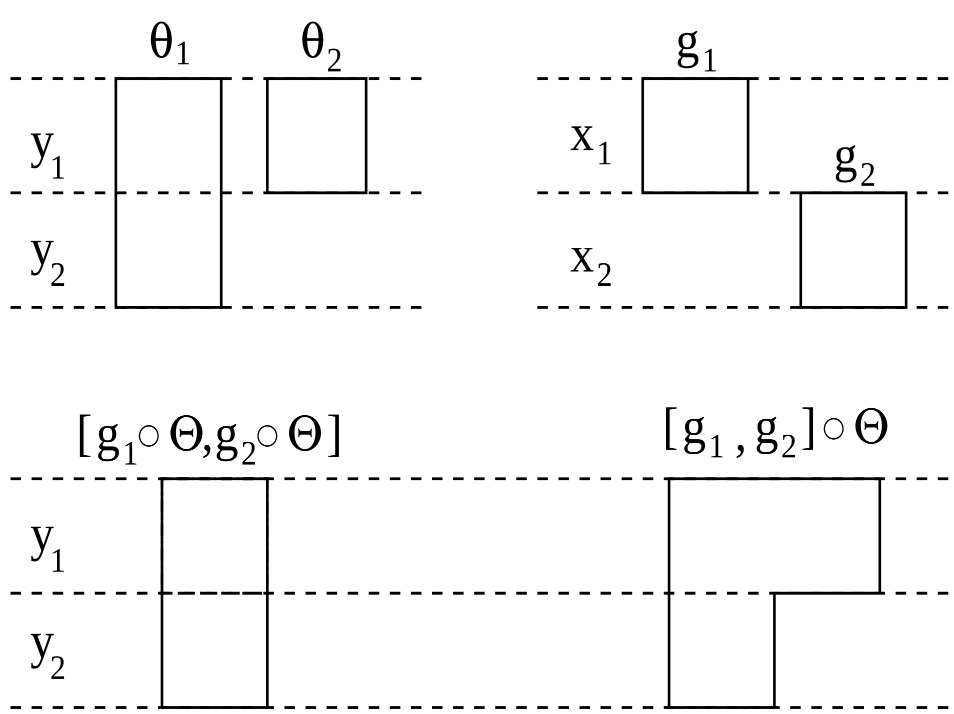
Notation:

$$g_1 \succ_y g_2$$

$$\Updownarrow$$

$$\forall \theta_i \; y | \theta_i \Rightarrow \mathsf{deg}_{x_i} g_1 \geq \mathsf{deg}_{x_i} g_2$$

Diagram:



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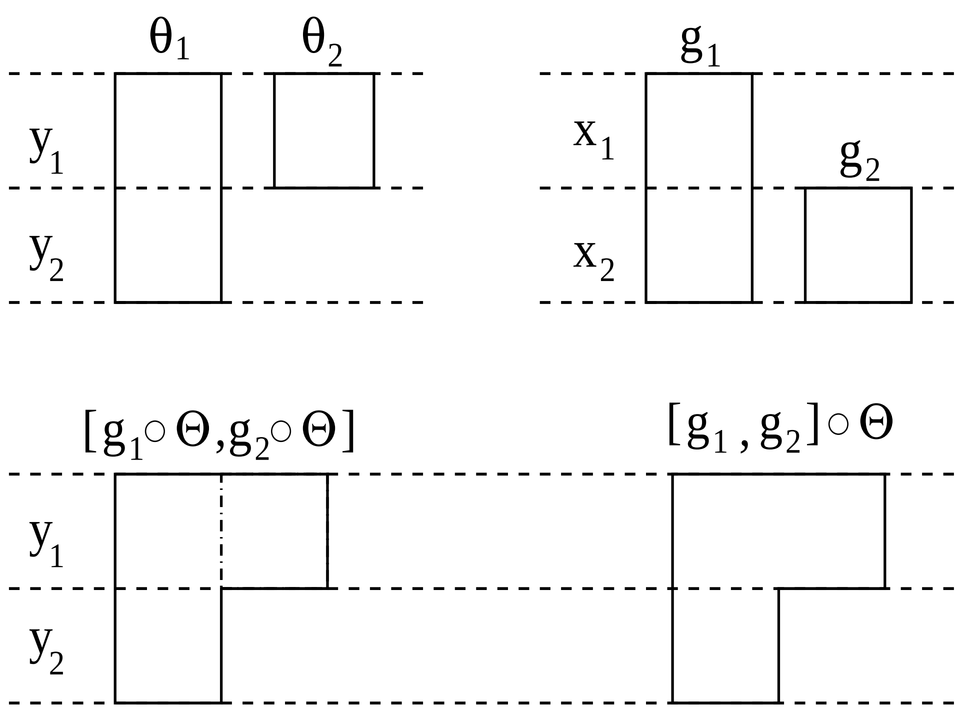

θ_1
θ_2
y_1
y_2
g_1
g_2
x_1
x_2
$[g_1 \circ \Theta, g_2 \circ \Theta]$
y_1
y_2
$[g_1, g_2] \circ \Theta$

Observe:

$\succ_y$ is a ***partial*** ordering

Observe:

$$\succ_y \text{ is a } \textbf{\textit{partial}} \text{ ordering}$$

$$\Downarrow$$

$$\exists p, q \text{ such that } p \not\succ q \wedge p \not\prec q$$

Fact:

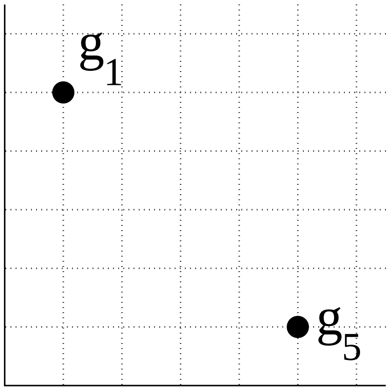
(Perry)

$$\mathsf{LCMCOMM}\left(g_i, g_j, \boldsymbol{\ominus}\right)$$

$$\Updownarrow$$

$$\forall y \quad g_i \succ_y g_j \vee g_i \prec_y g_j$$

Definition:



$$\mathsf{LCMPATH}\left(g_i, g_j, \ominus\right)$$

$$\Updownarrow$$

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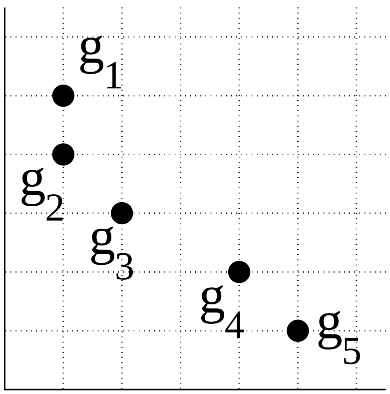

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$$\forall i \neq j \ \exists k_1, \ldots, k_s \ g_i = g_{k_1} \ g_j = g_{k_s}$$

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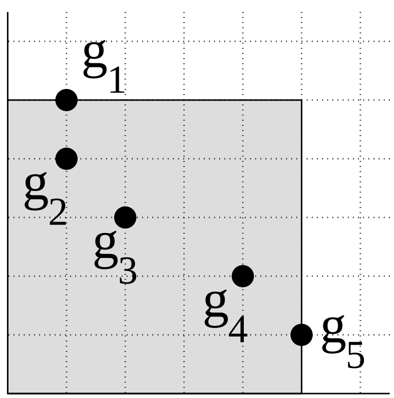

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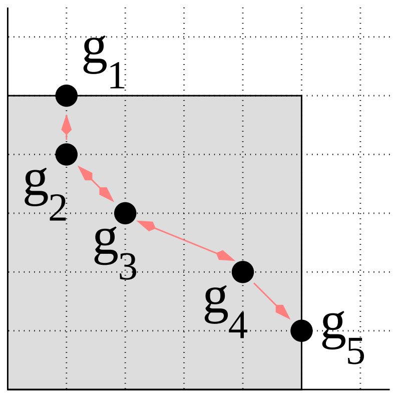

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$$\forall l \; \overline{g_{k_l}} \circ \overline{\ominus} | \left[\overline{g_i} \circ \overline{\ominus}, \overline{g_j} \circ \overline{\ominus}\right]$$
$$\wedge \mathsf{LCMCOMM}\left(g_{k_l}, g_{k_{l+1}}, \ominus\right)$$

Example:

$$G = \left\{ x^2yz + R_1, xyzw + R_2, xy^2w + R_3 \right\}$$

$$\Theta = (ab + \cdots, a + \cdots, c + \cdots, d + \cdots)$$

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$$\therefore \mathsf{LCMPATH}\,(g_1, g_3, \Theta)$$

Refinements of HHC ($m \leq n$)

Hong and Perry: $\forall G, \Theta$

$$\forall i \neq j \qquad \mathsf{LCMPATH}\left(g_i, g_j, \Theta\right)$$

$$\Downarrow$$

$$\forall G', \Theta' \qquad \overline{G} = \overline{G'} \quad \overline{\Theta} = \overline{\Theta'}$$

$$\mathsf{GB}\left(G'\right) \Rightarrow \mathsf{GB}\left(G' \circ \Theta'\right)$$

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$$\text{A. } \exists G_1, \Theta_1 \qquad \begin{array}{l} 1.\ \overline{G_1} = \overline{G} \wedge \overline{\Theta_1} = \overline{\Theta} \\ 2.\ \mathsf{GB}\,(G) \Rightarrow \mathsf{GB}\,(G \circ \Theta) \end{array}$$

Refinements of HHC ($m \leq n$)

Hong and Perry **(Conjecture):** $\forall G, \Theta$

$$\forall i \neq j \qquad \neg \mathsf{LCMPATH}\left(g_i, g_j, \Theta\right)$$

$$\Downarrow$$

$$\text{B. } \exists G_2, \Theta_2 \qquad 1.\ \overline{G_1} = \overline{G} \wedge \overline{\Theta_1} = \overline{\Theta}$$

$$2.\ \mathsf{GB}\,(G) \not\Rightarrow \mathsf{GB}\,(G \circ \Theta)$$

Example:

$$G = \left\{x^{20}y^{15}z^{10}, x^{15}y^{15}z^{2}w^{8}, x^{18}y^{30}w^{16}\right\}$$

$$\Theta = (ab + b + 1, b + c + 1, c - 1, d + 1)$$

HPC:

Current Maple:

Example:

$$G = \{x^{20}y^{15}z^{10}, x^{15}y^{15}z^{2}w^{8}, x^{18}y^{30}w^{16}\}$$

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HPC: 0.050 seconds

Current Maple: >57000 seconds $\approx$ 16 hours

>1,140,000 times faster!!!

Observation:

(Lazard)

$$\mathrm{LCMCOMM} \parallel \mathrm{BC1}$$

$$\mathrm{LCMPATH} \parallel \mathrm{BC2}$$

Composition (19b)

$\mathsf{LCMCOMM}\left(g_i, g_j, \ominus\right)$ Don't have to check $S_{ij}^{(\ominus)}$

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BC1 Don't have to check S_{ij}

Composition (19c)

$$\mathsf{LCMPATH}\left(g_i, g_j, \Theta\right) \quad \exists k_1, \ldots, k_s\ \overline{g_{k_i}} \circ \overline{\Theta} \vert \left[\overline{g_i}, \overline{g_j}\right]$$

Composition (19c)

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$$\mathsf{BC2} \qquad \exists k_1, \ldots, k_s\ \overline{g_{k_i}} |\ \left[\overline{g_i}, \overline{g_j}\right]$$

Question:

BC1, BC2 ***only*** pre-$\ominus$ CC?

Fact:

(Perry)

$$\overline{G} = \{x_1x_2, x_1x_3, x_1x_4\}$$

$$\Downarrow$$

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$$\Downarrow$$

$$\left.\begin{array}{c} \forall i \; g_i \text{ "reduced"} \\ S_{12} \longrightarrow 0 \wedge S_{23} \longrightarrow 0 \end{array}\right\} \Rightarrow S_{13} \longrightarrow 0$$

Fact:

$$\forall p_1, p_2, p_3 \qquad \left\{ \begin{array}{l} p_2 \not\| [p_1, p_3] \\ (p_1 p_3) \not\| (p_1, p_2) \wedge (p_1 p_3) \not\| (p_2, p_3) \end{array} \right.$$

$$\Downarrow$$

Fact:

$$\forall p_1, p_2, p_3 \qquad \left\{ \begin{array}{l} p_2 \not\parallel [p_1, p_3] \\ (p_1 p_3) \not\parallel (p_1, p_2) \wedge (p_1 p_3) \not\parallel (p_2, p_3) \end{array} \right.$$

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$$\exists g_1, g_2, g_3 \forall i \; \overline{g_i} = p_i \qquad (S_{12} \to 0 \wedge S_{23} \to 0) \Rightarrow S_{13} \to 0$$

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Why?

Let $G = \{p_1 + (p_1, p_2), p_2, p_3\}$

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$$\begin{aligned} S_{12} &= \frac{[p_1, p_2]}{p_1}(p_1, p_2) \\ &= \frac{p_1 p_2}{p_1 (p_1, p_2)}(p_1, p_2) \end{aligned}$$

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$S_{12} = p_2 \rightarrow 0$

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Conjecture:

$$\forall p_1, p_2, p_3 \qquad \begin{cases} p_2 \not\mid [p_1, p_3] \\ (p_1 p_3) \mid (p_1, p_2) \wedge (p_1 p_3) \mid (p_2, p_3) \end{cases}$$

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Conjecture:

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$$\forall g_1, g_2, g_3 \forall i \, \overline{g_i} = p_i \qquad (S_{12} \to 0 \wedge S_{23} \to 0) \Rightarrow S_{13} \to 0$$

Special Case: $\Theta = \overline{\Theta}$

Definition:

$$\triangle_{ij} = \frac{\left[\overline{g_i}, \overline{g_j}\right] \circ \ominus}{\left[\overline{g_i} \circ \ominus, \overline{g_j} \circ \ominus\right]}$$

Definition:

$$\blacktriangle\left(g_i, g_j, \ominus\right)$$

$$\Updownarrow$$

$$\exists h_k \, S_{ij} \longrightarrow 0 \ \wedge \ \forall k \, \blacktriangle_{ij}|h_k \circ \ominus$$

Refinement of HHC

(Perry)

$$\forall G, \ominus \quad \ominus = \overline{\ominus}$$

$$\forall i \neq j \, \Delta \left(g_i, g_j, \ominus\right)$$

$$\Downarrow$$

$$\mathsf{GB}\,(G) \Rightarrow \mathsf{GB}\,(G \circ \ominus)$$

Fact:

$$\mathsf{LCMPATH}\left(g_i, g_j, \boldsymbol{\ominus}\right) \Rightarrow \boldsymbol{\Delta}\left(g_i, g_j, \boldsymbol{\ominus}\right)$$

Fact:

$$\mathsf{LCMPATH}\left(g_i, g_j, \boldsymbol{\ominus}\right) \not\Leftarrow \boldsymbol{\Delta}\left(g_i, g_j, \boldsymbol{\ominus}\right)$$

Summary

1. Combinatorial Criteria $\rightarrow$ Time savings!!!

-

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2. Possibilities Pre- and Post- Composition

-

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3. Related work:

   - Hong and Minimair (Resultants)

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3. Related work:

   - Nordbeck (SAGBI bases)

Summary

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3. Related work:

   - Hong (Symmetric Polynomials), ...

Future Research

1. Prove remaining conjectures

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$$\frac{d}{dx} f\left(g\left(x\right)\right) = f'\left(g\left(x\right)\right) \cdot g'\left(x\right)$$

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2. “Chain rule” for Groebner Bases:

$$\frac{d}{dx} f\left(g\left(x\right)\right) = f'\left(g\left(x\right)\right) \cdot g'\left(x\right)$$

$$\mathcal{G}\left(G, \Theta\right) = \mathsf{GB}\left(G \circ \Theta\right)$$