Efficiently swapping columns in the Macaulay matrix

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Overview

1. From linear to non-linear algebra
2. Ordering columns of the Macaulay matrix
3. Finding orders efficiently
4. Conclusion

Outline

1. From linear to non-linear algebra
2. Ordering columns of the Macaulay matrix
3. Finding orders efficiently
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$$\begin{aligned}
x_1 \qquad\quad + 7x_3 - 3x_4 + 12x_5 &= 0\\
8x_1 \qquad\quad + 4x_3 + 3x_4 \qquad\quad &= 0\\
-2x_1 + 6x_2 \qquad\qquad\qquad\qquad &= 12\\
x_1 \qquad + 7x_3 \qquad\qquad\qquad &= 0
\end{aligned}$$

Questions:

- Do solutions exist?
- Dimension?
- etc.

Linear algebra

$$
\begin{aligned}
x_1 \qquad &+ 7x_3 - 3x_4 + 12x_5 &= 0\\
8x_1 \qquad &+ 4x_3 + 3x_4 &= 0\\
-2x_1 + 6x_2 & &= 12\\
x_1 \qquad &+ 7x_3 &= 0
\end{aligned}
$$

Vector space: add polynomials, multiply scalars

Rewrite as matrix:

$$
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 1 \end{array} \\
\left(\begin{array}{cccccc}
1 & & 7 & -3 & 12 & \\
8 & & 4 & 3 & & \\
-2 & 6 & & & & 12 \\
1 & & 7 & & &
\end{array}\right)
\end{array}
\begin{array}{c} \\ f_4 \\ f_3 \\ f_2 \\ f_1 \end{array}
$$

Solve? need “nice” form

Linear algebra

$$
\begin{array}{c c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 1 \end{array} & \\
\begin{pmatrix}
1 & & 7 & -3 & 12 & \\
8 & & 4 & 3 & & \\
-2 & 6 & & & & 12 \\
1 & & 7 & & &
\end{pmatrix} & \begin{array}{c} f_4 \\ f_3 \\ f_2 \\ f_1 \end{array}
\end{array}
$$

Vector space $\Longrightarrow$ Swap columns!

$$
\begin{array}{c c}
\begin{array}{cccccc} x_5 & x_4 & x_2 & x_3 & x_1 & 1 \end{array} & \\
\begin{pmatrix}
12 & -3 & & 7 & 1 & \\
 & 3 & & 4 & 8 & \\
 & & 6 & & -2 & 12 \\
 & & & 7 & 1 &
\end{pmatrix} & \begin{array}{c} f_4 \\ f_3 \\ f_2 \\ f_1 \end{array}
\end{array}
$$

Nonlinear algebra

$$xy - 1 = 0 \qquad x^2 + y^2 - 4 = 0$$

~~Vector space~~ Ideal: add, multiply polynomials

Nonlinear algebra

$$xy - 1 = 0 \qquad x^2 + y^2 - 4 = 0$$

$$\begin{array}{cc}
\begin{array}{cccccccccccc}
 & xy^3 & x^3 & x^2y & xy^2 & y^3 & x^2 & xy & y^2 & x & y & 1
\end{array} & \\
\left(\begin{array}{cccccccccccc}
\ddots & & & & & & & & & & & \\
 & & 1 & & 1 & & & & & -4 & & \\
 & & & 1 & & 1 & & & & & -4 & \\
 & & & & & & 1 & & 1 & & & -4 \\
\ddots & & & & & & & \ddots & & & & \\
 & 1 & & & & & & & -1 & & & \\
 & & & 1 & & & & & & -1 & & \\
 & & & & 1 & & & & & & -1 & \\
 & & & & & & & 1 & & & & -1
\end{array}\right) &
\begin{array}{c}
\vdots \\ xf_2 \\ yf_2 \\ f_2 \\ \vdots \\ y^2f_1 \\ xf_1 \\ yf_1 \\ f_1
\end{array}
\end{array}$$

Macaulay matrix

$$
\begin{array}{cc}
\begin{array}{cccccccccccc} & xy^3 & x^3 & x^2y & xy^2 & y^3 & x^2 & xy & y^2 & x & y & 1 \end{array} & \\
\left(\begin{array}{cccccccccccc}
\ddots & & & & & & & & & & & \\
 & & 1 & & 1 & & & & & -4 & & \\
 & & & 1 & & 1 & & & & & -4 & \\
 & & & & & & 1 & & 1 & & & -4 \\
\ddots & & & & & & & \ddots & & & & \\
 & 1 & & & & & & & -1 & & & \\
 & & & 1 & & & & & & -1 & & \\
 & & & & 1 & & & & & & -1 & \\
 & & & & & & & 1 & & & & -1
\end{array}\right) &
\begin{array}{c} \vdots \\ xf_2 \\ yf_2 \\ f_2 \\ \vdots \\ y^2f_1 \\ xf_1 \\ yf_1 \\ f_1 \end{array}
\end{array}
$$

Not "nice" form!

$$
\begin{array}{cc}
\begin{array}{ccccccccccc} xy^3 & x^3 & x^2y & xy^2 & y^3 & x^2 & xy & y^2 & x & y & 1 \end{array} & \\
\left(\begin{array}{ccccccccccc}
\ddots & & & & & & & & & & \\
 & 1 & & 1 & & & & & -4 & & \\
 & & 1 & & 1 & & & & & -4 & \\
 & & & & & 1 & & 1 & & & -4 \\
\ddots & & & & & & \ddots & & & & \\
1 & & & & & & & -1 & & & \\
 & & 1 & & & & & & -1 & & \\
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 & & & & & & 1 & & & & -1
\end{array}\right) &
\begin{array}{c} \vdots \\ xf_2 \\ yf_2 \\ f_2 \\ \vdots \\ y^2f_1 \\ xf_1 \\ yf_1 \\ f_1 \end{array}
\end{array}
$$

Can we swap?

$$\begin{pmatrix} x^2y & y^3 & 1 & \\ 1 & 1 & -4 & yf_2 \\ 1 & & -1 & xf_1 \end{pmatrix}$$

Can we swap?

$$\begin{array}{cc} \begin{matrix} x^2y & y^3 & 1 \end{matrix} & \\ \begin{pmatrix} 1 & 1 & -4 \\ 1 & & -1 \end{pmatrix} & \begin{matrix} yf_2 \\ xf_1 \end{matrix} \end{array}$$

Why not? consider $y > x$

$$\begin{array}{cc} \begin{matrix} y^3 & x^2y & 1 \end{matrix} & \\ \begin{pmatrix} 1 & 1 & -4 \\ & 1 & -1 \end{pmatrix} & \begin{matrix} yf_2 \\ xf_1 \end{matrix} \end{array}$$

Other rows non-nice

$$
\begin{array}{cc}
\begin{array}{ccccccccccc}
xy^3 & y^3 & xy^2 & x^3 & x^2y & y^2 & xy & x^2 & y & x & 1
\end{array} & \\
\left(\begin{array}{ccccccccccc}
\ddots & & & & & & & & & & \\
 & & 1 & 1 & & & & & & -4 & \\
 & 1 & & & 1 & & & & -4 & & \\
 & & & & & 1 & & 1 & & & -4 \\
\ddots & & & & & & \ddots & & & & \\
1 & & & & & -1 & & & & & \\
 & & & & 1 & & & & & -1 & \\
 & & 1 & & & & & & -1 & & \\
 & & & & & & 1 & & & & -1
\end{array}\right) &
\begin{array}{c}
\vdots \\ xf_2 \\ yf_2 \\ f_2 \\ \vdots \\ y^2f_1 \\ xf_1 \\ yf_1 \\ f_1
\end{array}
\end{array}
$$

To sum up

	structure	+	×	swap columns?
Linear	vec space	poly	const	yes
Nonlinear	ideal	poly	poly	no (not easily)

Why do we care?

row-echelon matrix	$\Longleftrightarrow$	vector basis
row-echelon Macaulay	$\Longleftrightarrow$	Gröbner basis

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  - easily analyze solutions
- applications
  - commutative algebra, algebraic geometry, differential equations, ...
  - astronomy, robotics, physics, biochemistry, ...
  - coding theory, cryptanalysis, ...

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- "nice" form of input system
  - easily analyze solutions
- applications
  - commutative algebra, algebraic geometry, differential equations, ...
  - astronomy, robotics, physics, biochemistry, ...
  - coding theory, cryptanalysis, ...
- "hard" to compute
  - worst case doubly exponential, average case "okay"
  - ***any tool to help is valuable!!!***

Outline

Questions

Can we swap columns in Macaulay's matrix while row reducing?

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YES!!!

(Gritzmann & Sturmfels, 1993; Caboara, 1993)

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How?

(ibid)

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How?

(ibid)

How, ***efficiently?***

Hmm.

How can we swap?

Definition

Admissible ordering:

- $\sigma \in \left(\mathbb{R}^{+}\right)^{n}$
- $\mathbf{x}^{a} < \mathbf{x}^{b} \quad \Longleftrightarrow \quad \sigma \cdot a < \sigma \cdot b$

How can we swap?

Definition

Admissible ordering:

- $\sigma \in \left(\mathbb{R}^{+}\right)^{n}$
- $\mathbf{x}^{a} < \mathbf{x}^{b} \iff \sigma \cdot a < \sigma \cdot b$

$\sigma = (4, 1)$

- $x > y^{3}$
- $x < y^{5}$

Important properties

- compatible with divisibility
  - $t \mid u \implies t \leq u$ $\quad$ $t \neq u$? then $t < u$
- compatible with multiplication
  - $t \leq u \implies tv \leq uv$
- well-ordering exists for any subset of monomials

Equivalent orderings

$$F = \{x^2 + y^2 - 4,\ xy - 1\}$$

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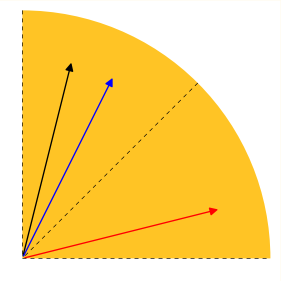
$$F = \{x^2 + y^2 - 4,\ xy - 1\}$$

Only two *effective* orderings: $x > y$ and $y > x$.



$$\begin{aligned}
(1,4) &\implies \operatorname{lm}(F) = \{y^2, xy\}\\
{\color{blue}(1,2)} &\implies \operatorname{lm}(F) = \{y^2, xy\}\\
{\color{red}(4,1)} &\implies \operatorname{lm}(F) = \{x^2, xy\}
\end{aligned}$$

Row reduction refines orderings

$$F = \{x^2 + y^2 - 4,\ xy - 1,\ y^3 + x - 4y\}$$

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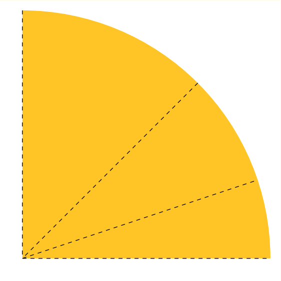
$$F = \{x^2 + y^2 - 4,\ xy - 1,\ y^3 + x - 4y\}$$

Cone splits! new ordering: $x > y^3$ but $x < y^4$



$$(1,4) \implies \mathrm{lm}(F) = \{y^2, xy, y^3\}$$
$$(3,2) \implies \mathrm{lm}(F) = \{x^2, xy, y^3\}$$
$$(4,1) \implies \mathrm{lm}(F) = \{x^2, xy, x\}$$

A vivid illustration

Sage program…

Some orderings better than others

$$F = \{x^2 + y^2 - 4,\ xy - 1\}$$

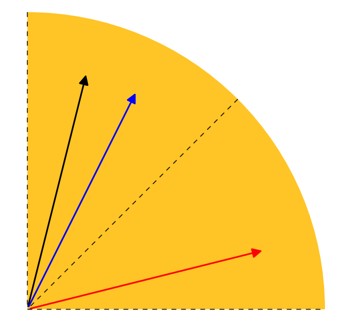
ordering	row operations	size of GB
$(4,1)$	5	4
$(1,2)$	4	3

Some orderings better than others

Caboara's results

system	standard row ops	standard size	"optimal" row ops	"optimal" size	rel. size row ops	rel. size size
random binomials	1340	239	80	40	6%	17%
zero-dimensional	29	13	26	7	90%	54%
"Morgenstern"	26	13	1	5	4%	38%
Cyclic-5	42	20	7	10	17%	50%
Katsura-4	24	22	25	23	104%	105%

Finding orders



Goal: find, compare orders in cone

Technique: linear programming! (details omitted)

“Optimal” ordering?

Metric: “Tentative” Hilbert function

intuitive: #mons of degree *d* *not* usable as pivots

precise: complicated! see “commutative algebra”

"Optimal" ordering?

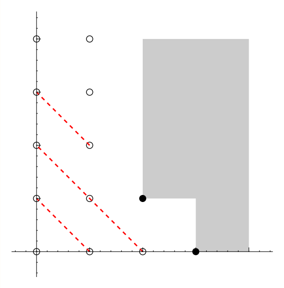
Metric: "Tentative" Hilbert function

intuitive: #mons of degree d *not* usable as pivots

precise: complicated! see "commutative algebra"

$$x^3 \quad , \quad x^2y$$

$$\begin{aligned} HF_I(0) &= 1 \\ HF_I(1) &= 2 \\ HF_I(2) &= 3 \\ HF_I(3) &= 2 \\ HF_I(4) &= 2 \\ &\vdots \end{aligned}$$



“Obvious” property

Reducing Macaulay decreases Hilbert

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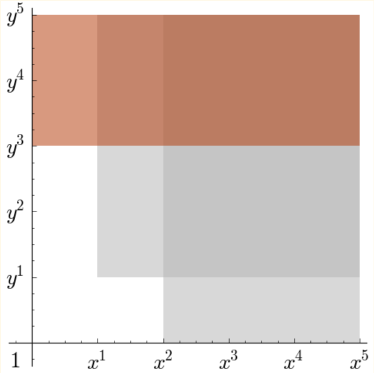
"Proof"



“Obvious” property

Reducing Macaulay decreases Hilbert

Strategy: Choose ordering that gives “smallest” HF.

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Bottleneck!

Many, *large* linear programs

- thousands, even millions of constraints
- solve many, many times
- crushes best linear solvers
- penalty outweighs benefits

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Many, *large* linear programs

- thousands, even millions of constraints
- solve many, many times
- crushes best linear solvers
- penalty outweighs benefits

Can we eliminate most constraints, attempts to solve?

Divisibility Criterion

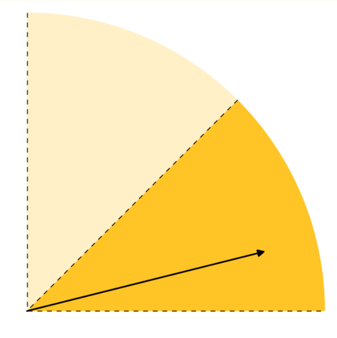
Recall ordering compatible with divisibility

- no constraints for divisors!

(Caboara, 1993)

Idea stay within current cone

- new constraints? only if consistent with old constraints

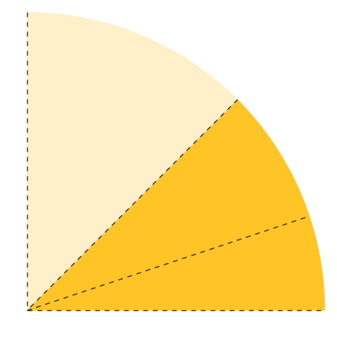


Committed to a cone?

Refining Criterion

Idea stay within current cone

- new constraints? only if consistent with old constraints



Refine in *current* cone

(Caboara, 1993)

Bottleneck persists!

- LPs still large, numerous
- successful refinement not *that* common
- in optimal cone? refinement useless

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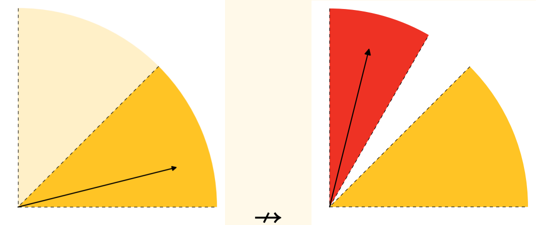
- in optimal cone? refinement useless

How can we shut refiner on, off… *reliably*?

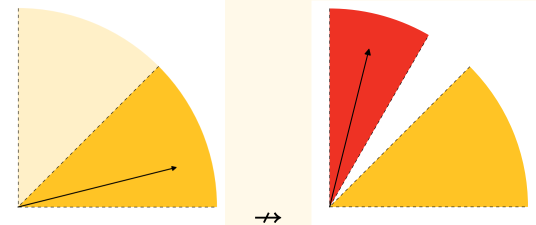
Disjoint Cones Criterion



LP fails?

- new, old constraints: disjoint cones
    - *remember new cone*
    - discard future LPs contained in new cone

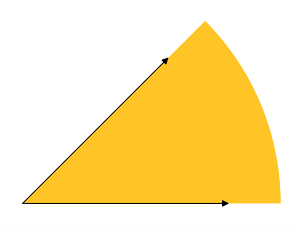
Disjoint Cones Criterion



LP fails?

- new, old constraints: disjoint cones
  - *remember new cone*
  - discard future LPs contained in new cone
- eliminates many LPs (not all)
- expensive, prefer to avoid

Boundary Vectors Criterion



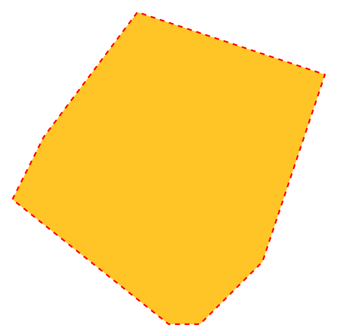
Theorem

Leading monomials compatible with ordering have greater weight wrt boundary vectors.

- Convexity
- Corner Point Theorem

Boundaries: $\geq 3d$

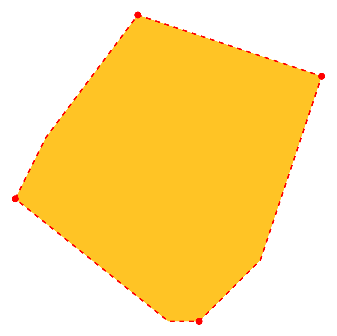
many corners, "hard" to find



(cross-section)

Boundaries: $\geq 3d$

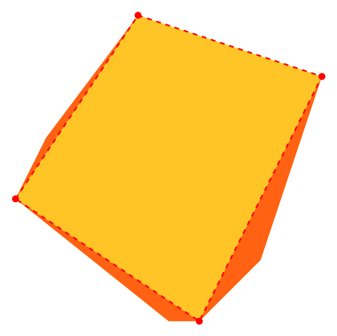
approximate: max, min vars



(cross-section)

Boundaries: underestimate

approximate: max, min vars



disallow some, allow only needed

Caboara's original examples

(Gröbner bases' sizes comparable to Caboara)

	linear programs				
	eliminated by:		computed by:		relative
system	disjoint	boundary	old	new	size
random binomials	0	22	25	10	40%
zero-dimensional	0	64	20	10	50%
"Morgenstern"	0	81	6	19	316%
Cyclic-5	0	379	327	16	5%
Katsura-4	5	439	108	37	34%

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"Morgenstern"?!?

- approximate boundary vectors exclude good monomial(s)
- compute larger basis
    - more polys $\Longrightarrow$ more mons $\Longrightarrow$ more LPs
    - still smaller, more efficient than static

Other examples

linear programs

system	eliminated by: disjoint	eliminated by: boundary	computed by: old	computed by: new	relative size
Cyclic-6	0	4,080	2,800	58	2%
Cyclic-7	12	134,158	*	145	N/A
Cyclic-6 homog	0	1,460	1,233	25	2%
Cyclic-7 homog	0	62,706	*	38	N/A

*terminated after growing to 1,000 constraints

But is it *efficient?*

“Underestimate”: max, min vars

- $O(n)$ space
- $O(n)$ time: change objective function, re-solve simplex
- Benefits far outweigh costs

Practically expensive?

Profiler data: Cyclic-6

Expensive routines: standard GB bottlenecks

- row reduction
  ~ 50% time
- identifying rows for reduction
  ~ 33% time

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Expensive routines: standard GB bottlenecks

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Inexpensive routines: dynamic techniques!

- computing orderings (simplex)
  ~ 4% time
- computing boundary vectors (simplex)
  ~ .04% time
- applying boundary vectors
  ~ .4%

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Observations

- Can swap columns, limited by
  - admissible orderings
  - previous orderings (Refining Criterion)
- Shut refiner off w/“high” certainty
- Restart refiner w/“high” certainty

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Boundary vectors provide
effective, efficient technique
to identify orderings

Questions / future work

- Better boundary vectors?
  - avoid “Morgenstern” penalty
- Other metrics of “good” basis?
  - size of reduction matrix? (more monomials?)
- Interaction w/ strategies for computation?
  - signature?
- Parallelism?

Finis

Thank you!

- LaTeX
  - LyX
  - Beamer
- sage
- People & mailing lists
  - Massimo Caboara
  - Nathann Cohen
  - Bernd Sturmfels
  - Help-GLPK, CBC digest, sage-support, sage-devel