When can we skip S-polynomial reduction?

(3 polynomials)

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8th September 2004

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Partially supported by NSF 53344

Motivation

- Gröbner bases $\longrightarrow$ S-polynomial reductions to 0
- Reduction: computationally expensive
- So, we would like to skip whenever possible
- *WHEN?*

When can we skip S-polynomial reduction?

S-polynomials

Definition:

$$\mathbf{S}_{\succ}(f,g) = \frac{\mathbf{lcm}\left(\mathbf{lt}_{\succ}(f), \mathbf{lt}_{\succ}(g)\right)}{\mathbf{lm}_{\succ}(f)} \cdot f - \frac{\mathbf{lcm}\left(\mathbf{lt}_{\succ}(f), \mathbf{lt}_{\succ}(g)\right)}{\mathbf{lm}_{\succ}(g)} \cdot g$$

Reduction

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$$g = h{\color{blue}f} + r$$

$$\downarrow$$

$$g \longrightarrow_{\color{blue}f}^{*} r$$

Reduction

"Get remainder"

$$g = h_1 f_1 + h_2 f_2 + h_3 f_3 + r$$

$$\downarrow$$

$$g \longrightarrow_F^* r$$

where $F = (f_1, f_2, f_3)$

Question, restated

When can we skip S-polynomial reduction?

Previous Results

- Buchberger, 1965

$$\mathbf{gcd}\left(\widehat{f}_1, \widehat{f}_3\right) = 1 \quad \Rightarrow \quad \mathbf{S}_{13} \rightarrow_F^* 0 \qquad \text{(BC1)}$$

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  $$\Rightarrow \quad \left[\mathbf{S}_{12} \rightarrow_F^* 0 \wedge \mathbf{S}_{23} \rightarrow_F^* 0 \Rightarrow \mathbf{S}_{13} \rightarrow_F^* 0\right] \qquad \text{(BC2)}$$

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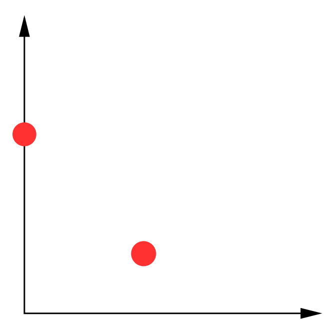

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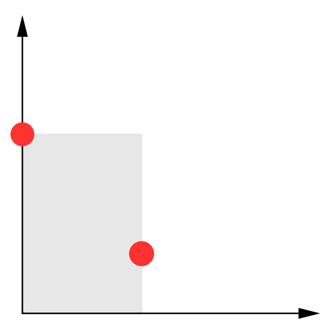

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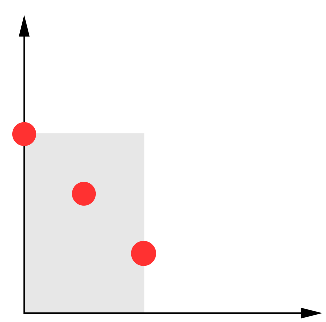

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Note:

- (BC1) $\Rightarrow$ Can skip $\mathbf{S}_{13}$
- (BC2) $\Rightarrow$ Can skip $\mathbf{S}_{13}$

Theorem (2004)

$$\boxed{\text{Can skip } \mathrm{S}_{13}}$$

$$\Updownarrow$$

$$\boxed{\text{CC}(t_1, t_2, t_3) := \text{CC1}(t_1, t_2, t_3) \text{ and } \text{CC2}(t_1, t_2, t_3)}$$

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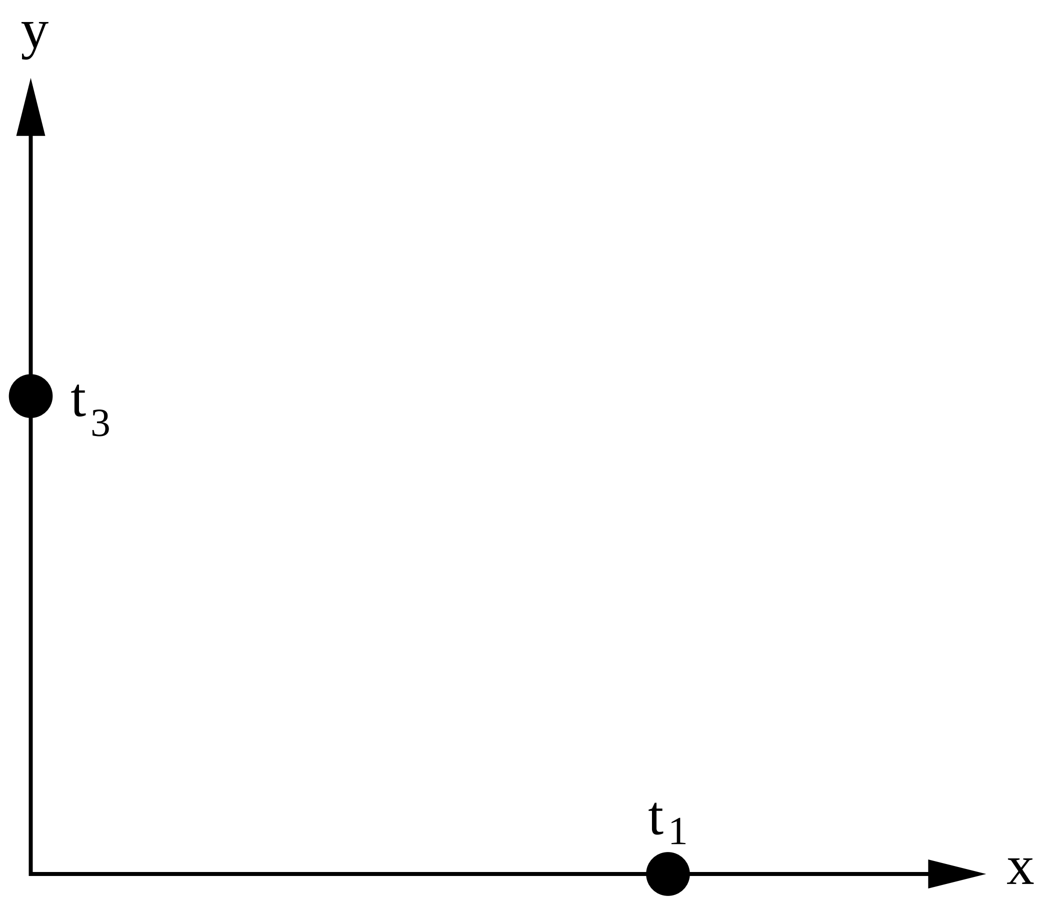
$\text{CC2}(t_1, t_2, t_3) \quad \Leftrightarrow \quad$ variable-wise: BC1 or BC2

Geometrically speaking

Where t_2?

$$t_1 = x^4 \qquad t_3 = y^3$$



All variables relatively prime

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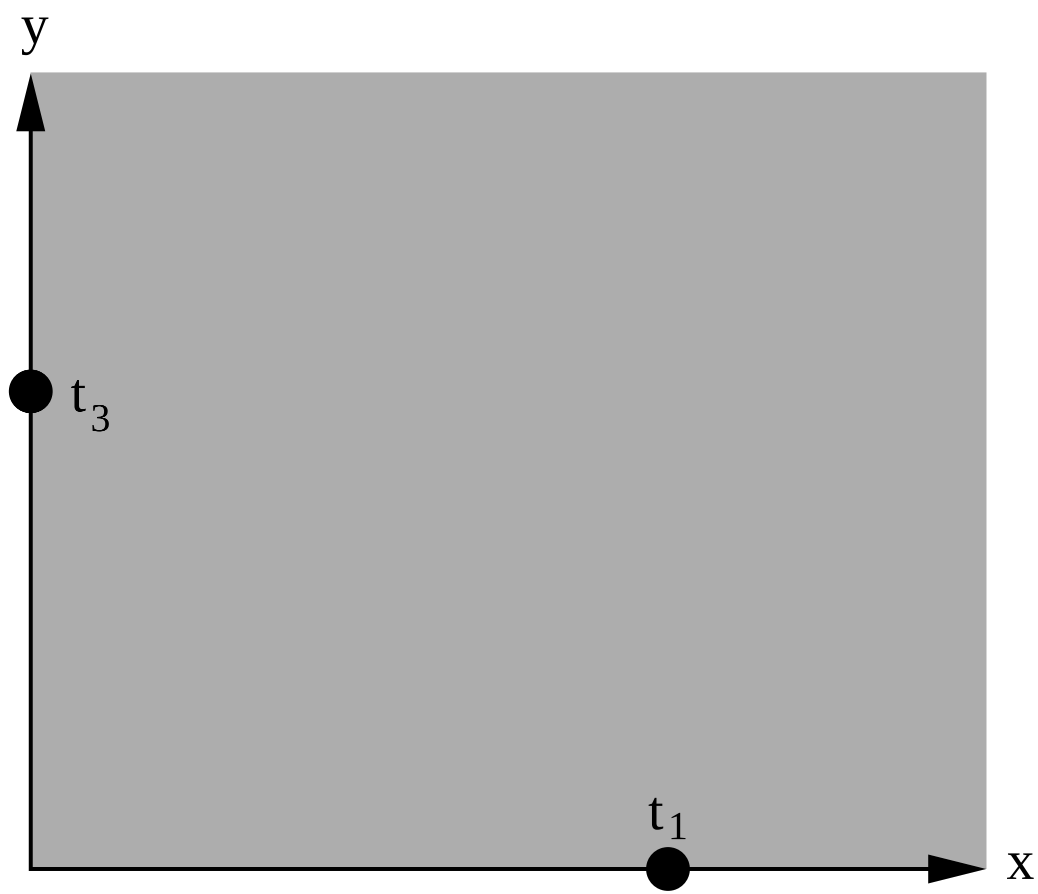

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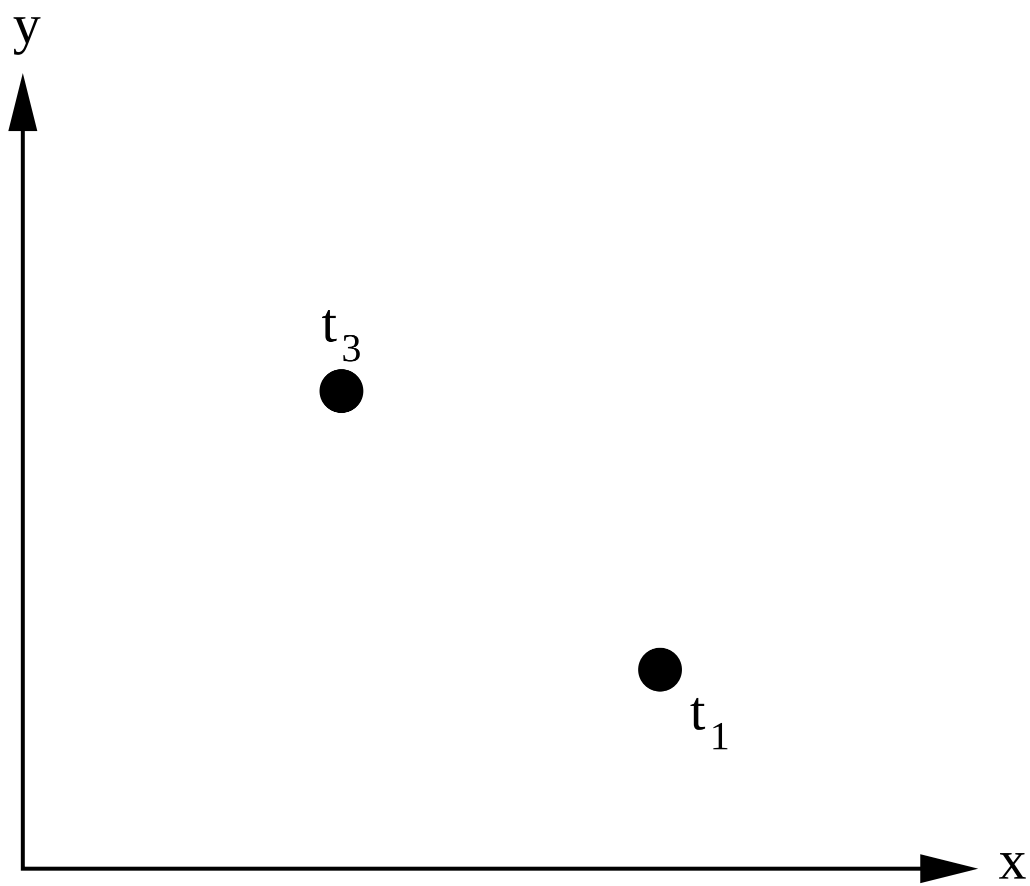

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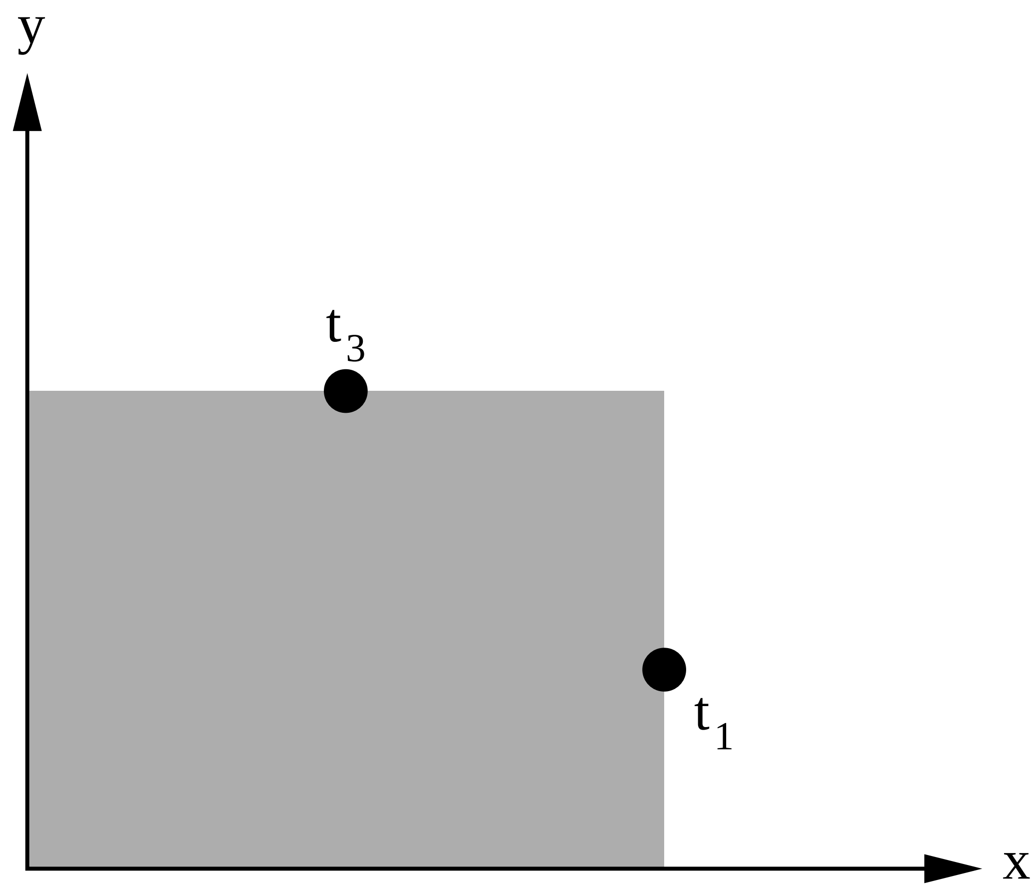

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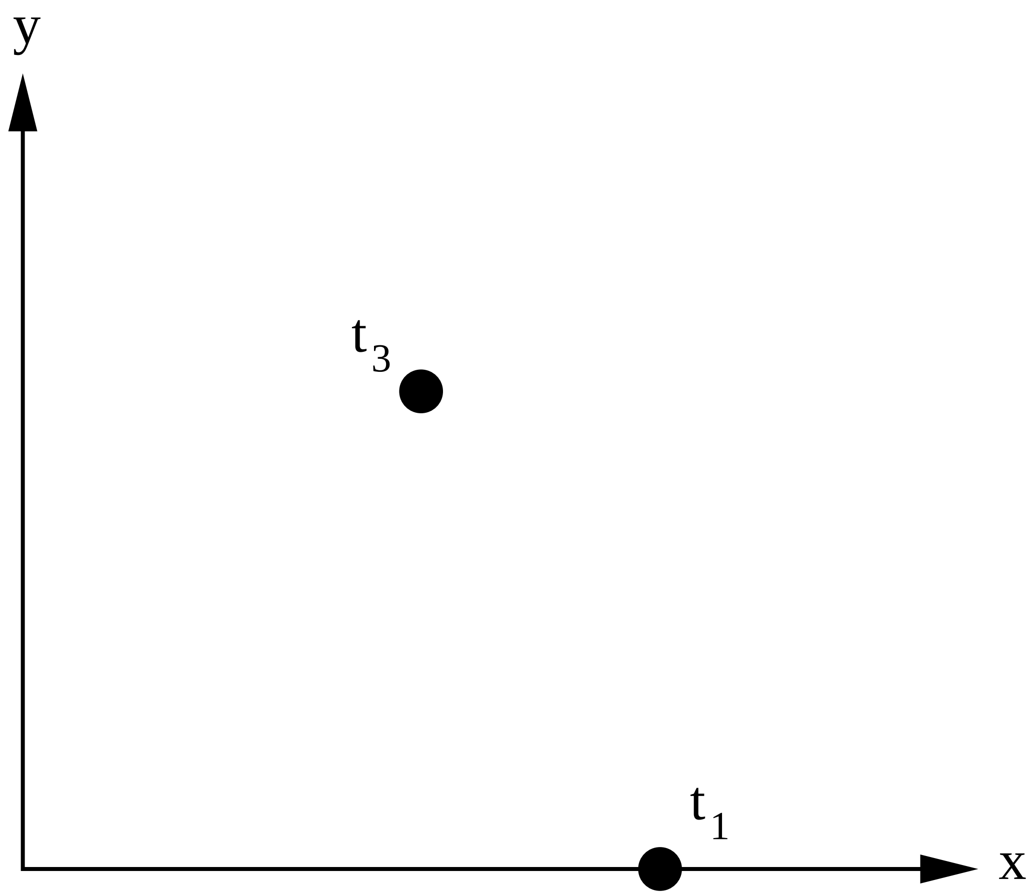

Relatively prime on y

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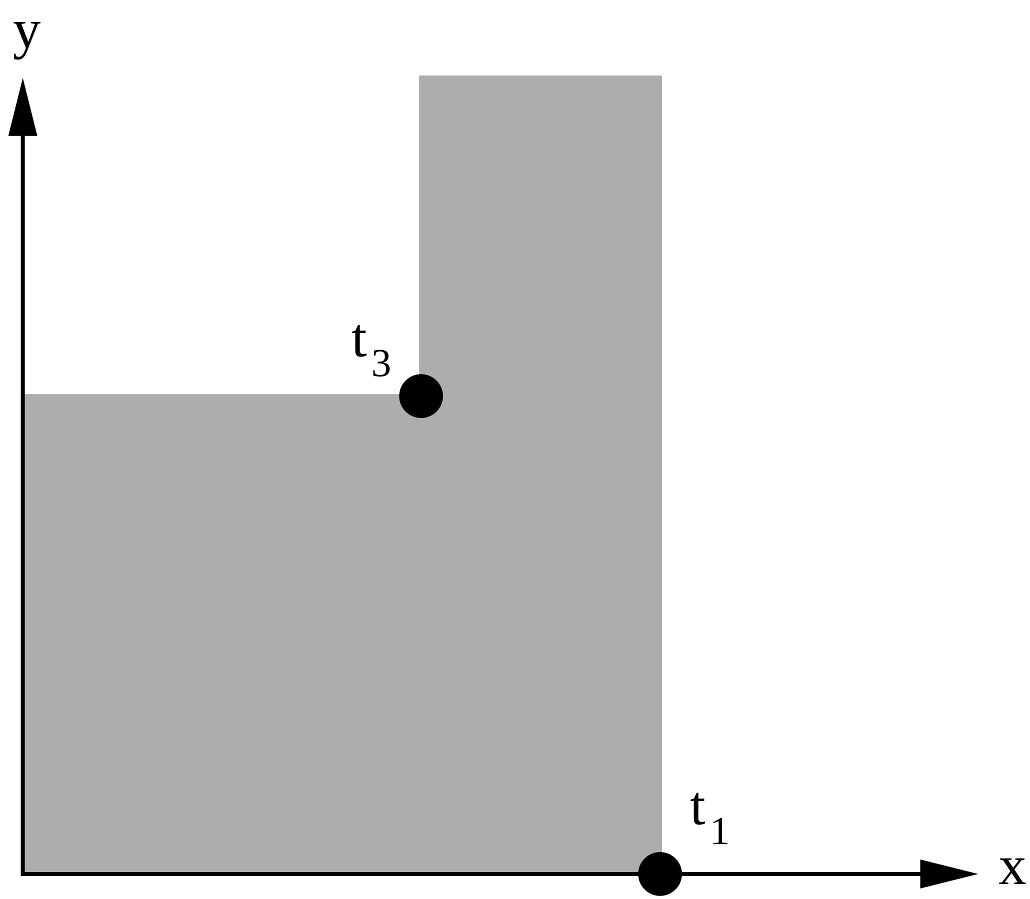

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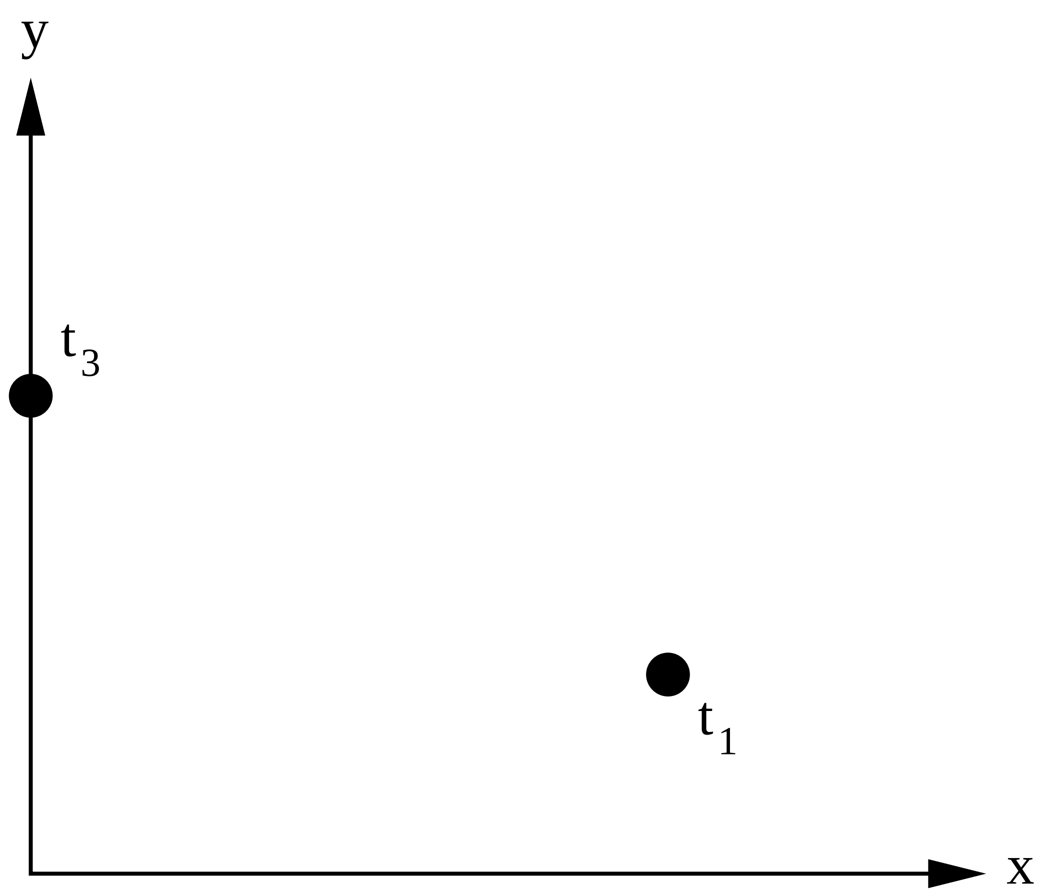

Relatively prime on x

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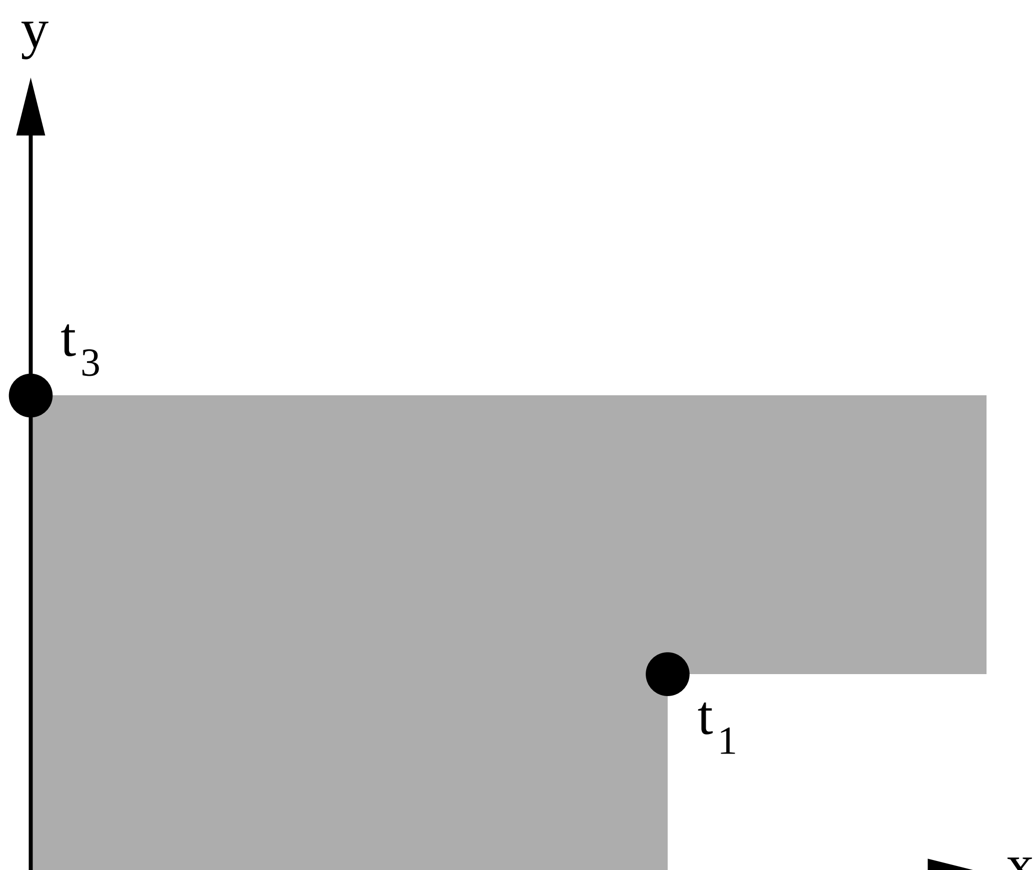

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- Finding minimal generating set for syzygy is *not* good enough

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$$\mathbf{S}_{12} \leftrightarrow \begin{pmatrix} y \\ -\ x^2 \\ 0 \end{pmatrix} \qquad \mathbf{S}_{23} \leftrightarrow \begin{pmatrix} 0 \\ z \\ -y^2 \end{pmatrix} \qquad \mathbf{S}_{13} \leftrightarrow \begin{pmatrix} z \\ 0 \\ xy \end{pmatrix}$$

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- We truly found *new* cases

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Contrapositive: $\neg$CC $\Rightarrow$ $\neg$Can skip $\mathbf{S}_{13}$

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$\neg$CC1:

$$f_1 = t_1 + \mathbf{gcd}\,(t_1, t_2) \quad f_2 = t_2 \quad f_3 = t_3$$

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$\neg$CC2:

$$f_1 = t_1 + u \quad f_2 = t_2 \quad f_3 = t_3$$

where

$$\forall x \quad \deg_x u = \begin{cases} \deg_x t_3 & x \neq y \\ \max\left(0, \deg_x \frac{t_1 t_3}{t_2}\right) & x = y \end{cases}$$

where $\deg_y t_2 > \deg_y \mathbf{lcm}\,(t_1, t_3)$

CC $\Rightarrow$ Can skip S_{13}

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We have eliminated $\forall$!

Summary

"When can we skip S-polynomial reduction?"

- Complete answer for three polynomials
- Four polynomials: ...?

Thank you!

Examples

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