Geometry of Hilbert polynomials and Gröbner bases

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Algebra is merely geometry in words;
geometry is merely algebra in pictures.
— Sophie Germain

Outline

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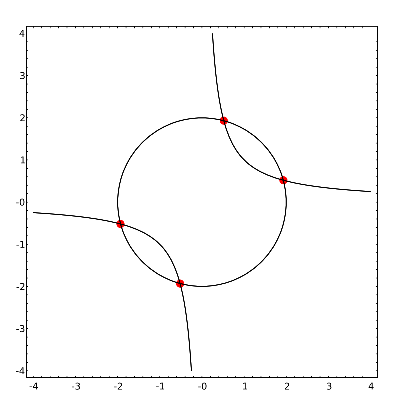
1. Gröbner bases and monomial orderings
2. Hilbert polynomials
3. Caboara, Sturmfels' algorithms
4. Future

Gröbner basis: definition from example



what can I say about ● ?

Monomial orderings

$$x^2 + xy^2$$

$$\swarrow \qquad \searrow$$

$x^2 + xy$ (lex) $\qquad$ $x^2 + xy^2$ (tdeg)

Remark

- notation: $\text{lm}(p)$
- uncountably many orderings
- given an ideal, finitely many equivalence classes

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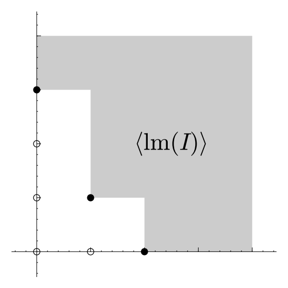
- given an ideal, finitely many equivalence classes

Gröbner basis: precise definition

G a **Gröbner basis** of ideal I?



- $I = \langle G \rangle \ldots$
-

Gröbner basis: precise definition

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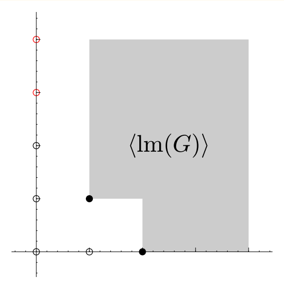

$$y^3 \notin \langle \mathrm{lm}(G) \rangle$$

- $I = \langle G \rangle$ but
- $\langle \mathrm{lm}(I) \rangle \neq \langle \mathrm{lm}(G) \rangle$

Gröbner basis: precise definition

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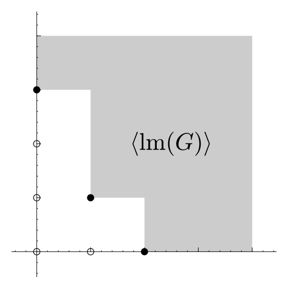


$y^3 \in \langle \mathrm{lm}(G) \rangle$

- $I = \langle G \rangle$ and
- $\langle \mathrm{lm}(I) \rangle = \langle \mathrm{lm}(G) \rangle$

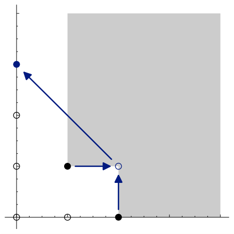
Gröbner basis: facts

- finite GB exists for any I
- GB varies by equivalence class
    - size, density, complexity
    - unique “reduced GB”

Computing Gröbner bases

Buchberger's algorithm (1965), others

- while $\langle \mathrm{lm}(G) \rangle \neq \langle \mathrm{lm}(I) \rangle$:



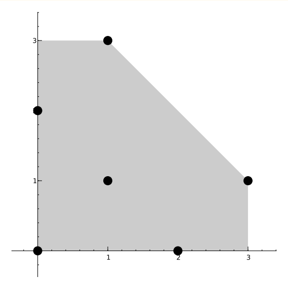
add ● to G

- ● "easy" iff $\langle \mathrm{lm}(G) \rangle \neq \langle \mathrm{lm}(I) \rangle$

Minkowski sum of Newton polyhedra

another view of orderings

$$F = \{x^2 + y^2 - 4, xy - 1\}$$



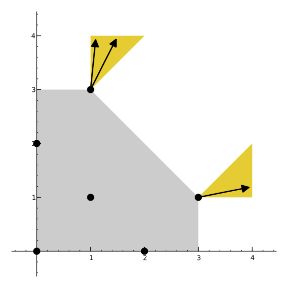
(vertex $\leftarrow$ exponent vector of each $u \in f_1 \cdots f_m$)

Big-time fact

Normal cones of Minkowski sum of F

$\updownarrow$

equivalence classes



classification of orderings $\longleftrightarrow$ discrete geometry

(Gritzmann and Sturmfels, 1993)

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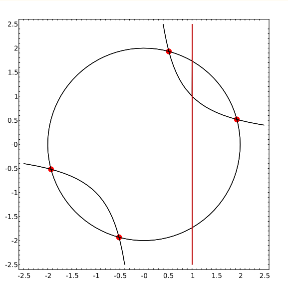
Hilbert function: definition from example



$\dim_{\mathbb{F}}(\bullet)$?

Hilbert function: precise definition

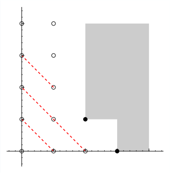
$R = \mathbb{F}[x_1, \ldots, x_n]$, homogeneous ideal

Hilbert function:

$$\begin{aligned} \mathrm{HF}_I : \mathbb{N} &\to \mathbb{N} \\ d &\to \dim_{\mathbb{F}} (R_d / I_d). \end{aligned}$$

Facts

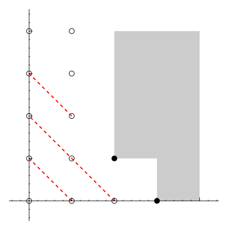
- $\exists\, \mathrm{HF}_I$ for all I
- $\mathrm{HF}_I = \mathrm{HF}_{\langle \mathrm{lm}(I) \rangle}$



$$\dim_{\mathbb{F}} (R_d / I_d) = \dim_{\mathbb{F}} \left(R / \langle \mathrm{lm}(I) \rangle \right)_d$$

Hilbert polynomial:

$$\mathrm{HP}_I(d) = \mathrm{HF}_I(d) \text{ for “sufficiently large” } d.$$



Facts

- $\exists\, \mathrm{HF}_I, \mathrm{HP}_I$ for all I
- $\deg \mathrm{HP}_I = \dim_{\mathbb{F}}(\bullet)$

$\mathrm{HF}_I(d) \leftrightsquigarrow \mathrm{GB}(I)$

Know $G = \mathrm{GB}(I)$, not HF_I?

- $\mathrm{HF}_{\langle \mathrm{lm}(G) \rangle} = \mathrm{HF}_I$
- "easy" to compute:
  - HS_I, power series expansion of HF_I
  - HP_I

Know HF_I, not $\mathrm{GB}(I)$?

- $\mathrm{HF}_I(d)$ degree d mononomials not in $\langle \mathrm{lm}(G) \rangle_d$?
  - ~~degree d pairs~~

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Problem statement

- GB varies by equivalence class
  - size, density, complexity
  - some orderings more efficient than others

Is it advisable to change orderings while computing a Gröbner basis?

“Tentative” Hilbert functions

$$J = \left\langle \mathrm{lm}\left(G^h\right)\right\rangle$$

Definition

tentative Hilbert function of $\langle G \rangle$ is HF_J

Example 1

- $G^h = \{x^2 + y^2 - 4h^2, xy - h^2\}$
- $\mathrm{lm}\left(G^h\right) = \{x^2, xy\}$

$$\mathrm{HS}_J(t) = \frac{t^2 - t - 1}{-t^2 + 2t - 1} \qquad \mathrm{HP}_J(d) = d + 2$$

$$\mathrm{HS}_I(d) = \frac{t^2 + 2t + 1}{-t + 1} \qquad \mathrm{HP}_I(d) = 4$$

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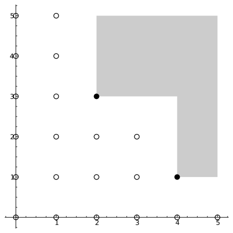
Example 2

$$G = \{x^3y^2 + 5776x^2y^3 + \cdots, x^4y + \cdots\}$$

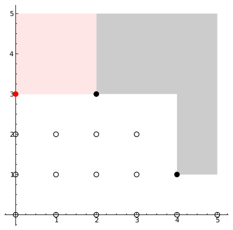
tdeg, $x > y$	tdeg, $x < y$
$\mathrm{HP}_J(d) = 4d - 1$	$\mathrm{HP}_J(d) = 3d + 4$

Which ordering “better”?

If G *not* GB,



$R/\langle \mathrm{lm}(G) \rangle$



$R/\langle \mathrm{lm}(I) \rangle$

$$\mathrm{HF}_I(d) = \dim_{\mathbb{F}} (R/I)_d$$

$\therefore$ minimize $\mathrm{HF}_{\langle \mathrm{lm}(G) \rangle}(d)$ *in long run*

Example

	d							
HP_I	0	1	2	3	4	5	6	#GB
$3d+4$	4	7	10	13	16	19	22	5
$4d-1$	-1	3	7	11	15	19	23	15

Gritzmann-Sturmfels algorithm

"Dynamic" Buchberger algorithm:

- after adding new polynomial:
  - compute HF for each normal cone
  - select ordering w/"best" HF

Gritzmann-Sturmfels, 1993

Caboara algorithm

Gritzmann-Sturmfels algorithm:

- compute HF for each normal subcone

Observations

- trade-off in complexity
  - most timings improve
  - some worsen
- only implementation of Gritzmann-Sturmfels algorithm?

Caboara, 1993

Caboara algorithm

Gritzmann-Sturmfels algorithm:

- compute HF for each normal subcone

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Caboara, 1993

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Open question

- small penalty to compute HF
- diminishing benefit from “better” order
- when quit computing HF?
- guess: when $\text{HF}_J(d)$ stable up to largest degree pair?

Finis

Thank you!

- LaTeX
  - LyX
  - Beamer
- sage
- audience