Gröbner Basis Detection for Two Polynomials

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Detection

Given f_1, f_2 exists $\succ$ $\mathbf{GB}_{\succ}(f_1, f_2)$?

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(If so, *find* $\succ$)

Example 1

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Yes!

All $\succ$

$\mathbf{lt}_{\succ}(f_1), \mathbf{lt}_{\succ}(f_2)$ relatively prime

Example 2

Given f_1, f_2 exists $\succ$ $\mathbf{GB}_{\succ}(f_1, f_2)$?

$$f_1 = x^6 + 1$$

$$f_2 = x^4y^3 - x^2y^3 + y^3 + x^5 + x^4 - x^3 - x^2 + x + 1$$

State of the Art

1993	Gritzmann, P. and Sturmfels, B.
1997	Sturmfels, B. and Wiegelmann, M.
2000	Gatermann, K.
2004	Hong, H. and Perry, J. (this talk)

Motivation

- Term ordering often irrelevant in applications
  (Any term ordering is fine, as long as there is one)

- Conversion of term orders often more efficient
  (If already Groebner basis, we want to know it)

Hong-Perry, 2004

Theorem:

$$\boxed{\mathrm{GB}_{\succ}(f_1, f_2)}$$

$$\Updownarrow$$

$$\boxed{\begin{array}{c}\mathbf{lt}_{\succ}(c_1), \mathbf{lt}_{\succ}(c_2) \text{ relatively prime} \\ \text{where } c_1, c_2 \text{ cofactors of } \mathrm{gcd}(f_1, f_2)\end{array}}$$

Hong-Perry, 2004

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Example:

$$f_1 = x(x+1)$$
$$f_2 = x\left(x^2+y+1\right)$$

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Example:

$$f_1 = x(x+1)$$
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$$\succ = \mathrm{tdeg}(x > y)$$

Hong-Perry, 2004

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Example:

$$f_1 = x(x + 1)$$

$$f_2 = x\left(x^2 + y + 1\right)$$

$$\succ = \mathrm{lex}(y > x)$$

Applying the theorem

For $t_1 \in c_1$ and $t_2 \in c_2$

If t_1 and t_2 relatively prime

If $\exists \succ$ s.t. $t_1 = \mathbf{lt}_\succ(c_1)$, $t_2 = \mathbf{lt}_\succ(c_2)$

Output *"Yes, > is a term ordering"*

Output *"No term ordering"*

Example 2

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$$g = x^4 - x^2 + 1$$

$$c_1 = x^2 + 1$$
$$c_2 = x + y^3 - 1$$

$$\left(x^2, x\right) \quad \left(x^2, y^3\right) \quad \left(x^2, 1\right) \quad (1, x) \quad \left(1, y^3\right) \quad (1, 1)$$

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$$c_2 = x + y^3 - 1$$

$$\left(x^2, y^3\right)$$

YES!

total degree

Comparison

Gritzmann-Sturmfels

- Minkowski sum of Newton polytopes; outer normal cones at each vertex
- several S-polynomial reductions
- works on n polynomials

Hong-Perry

- one gcd computation
- several systems of inequalities
- works on 2 polynomials

Acknowledgements

Erich Kaltofen
Dan Finkel
Jill Reese
Scott Pope

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Appendix

Needed lemma

Let m be a vector obtained as above.
Then $\exists i$ s.t.

$$M = \begin{pmatrix} m \\ e_1 \\ \vdots \\ e_{i-1} \\ e_{i+1} \\ \vdots \\ e_{n-1} \end{pmatrix}$$

is a representation of a term ordering that extends m.

Gritzmann-Sturmfels, 1993

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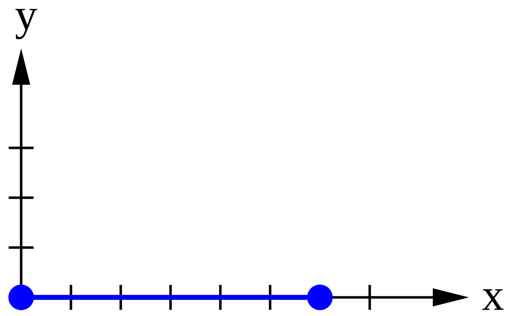
3. # equivalence classes finite *(Mora, Robbiano)*

Gritzmann-Sturmfels, 1993

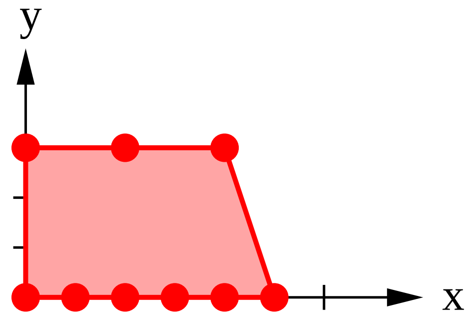
1. Brute force: try all term orderings one-by-one
2. Some term orderings "equivalent" wrt f_1, f_2
3. # equivalence classes finite *(Mora*, *Robbiano)*
4. classes $\leftrightarrow$ "open normal cones" of Minkowski sum

Example 2 revisited

$f_1 = x^6 + 1$



$f_2 = x^4y^3 - x^2y^3 + y^3 + \ldots$



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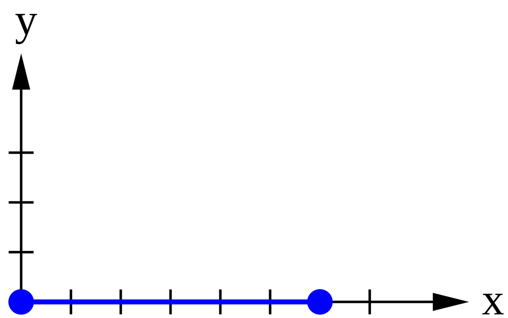
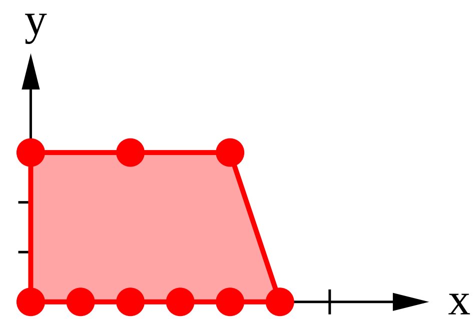
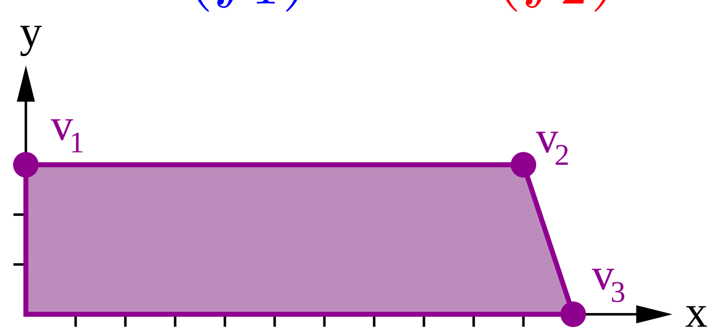




$$N\left(f_1\right) + N\left(f_2\right)$$



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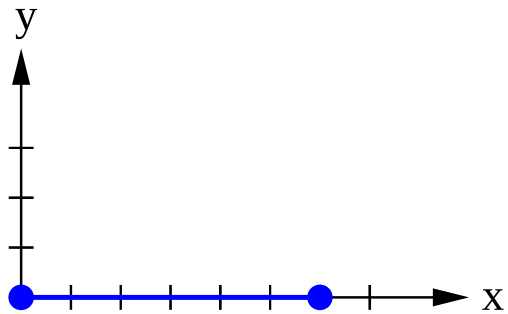
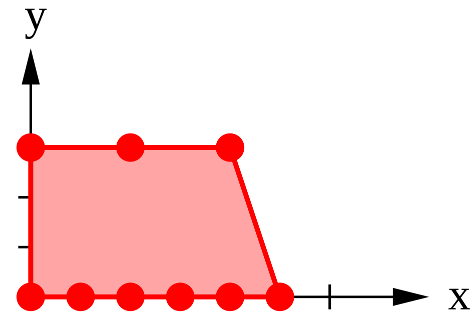
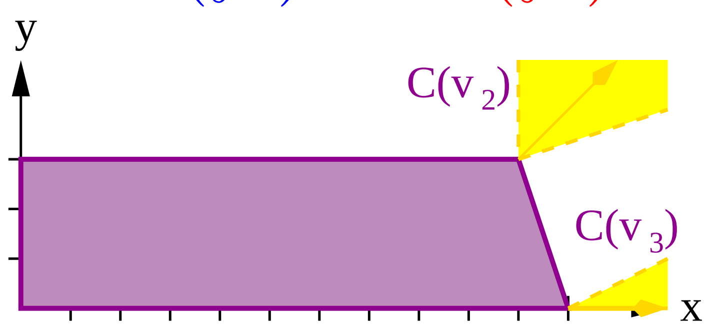




$$N(f_1) + N(f_2)$$



Candidates:

$$\langle 1, 0 \rangle \in C(v_3) \qquad \langle 1, 1 \rangle \in C(v_2)$$

Example 2 revisited

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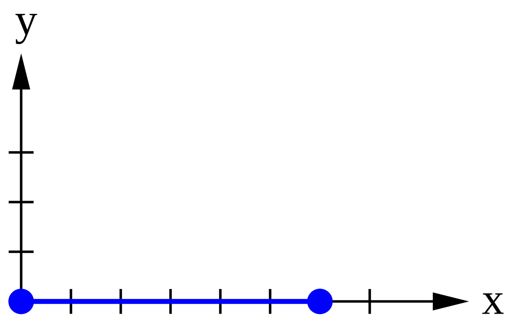
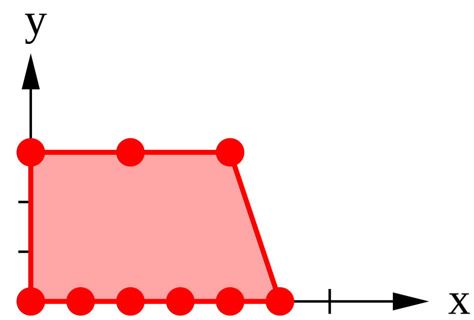
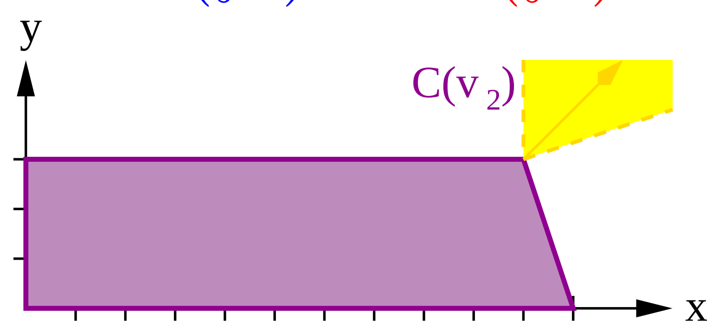




$$N(f_1) + N(f_2)$$



Solution:

$$\langle 1, 1 \rangle \in C(v_2) \quad \Rightarrow \quad M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

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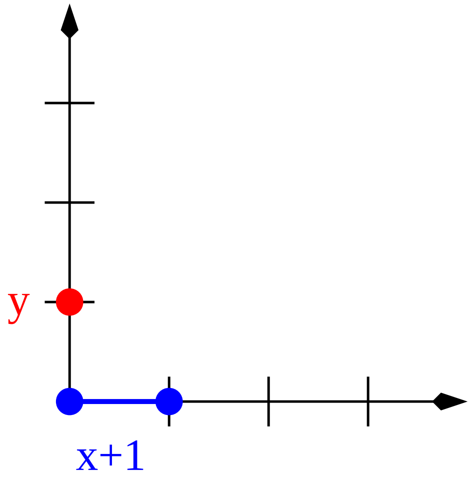

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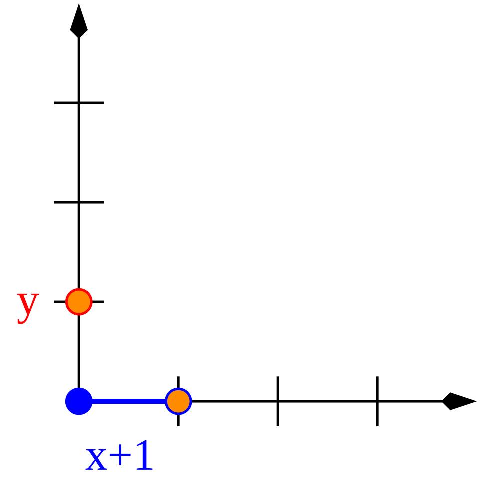

$$\therefore \forall_{\succ}\ \mathbf{GB}_{\succ}(f_1, f_2)$$

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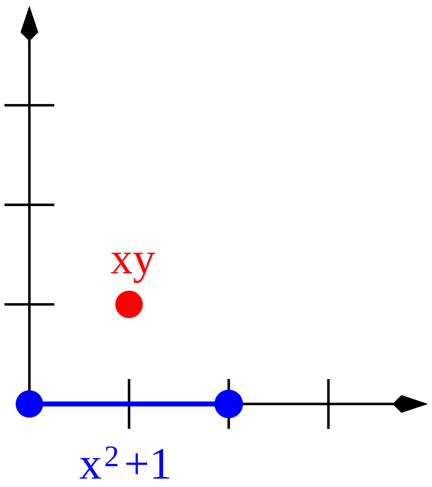

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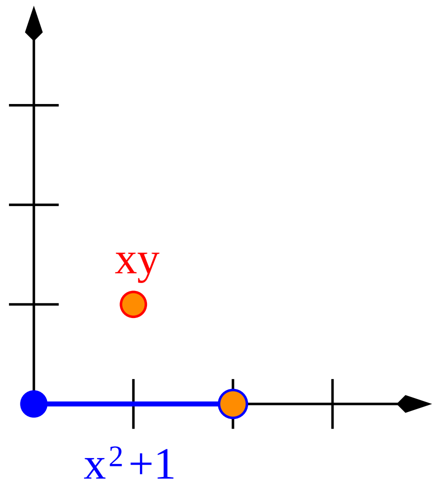

$\therefore$ NO $\succ$ for $\mathbf{GB}_{\succ}(f_1, f_2)$