Nim$^\infty$

or, *From the ridiculous to the sublime is but a step.*
(with apologies to Napoleon)

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Department of Mathematics
University of Southern Mississippi

March 22, 2013

Overview

Background

The Hilbert-Dickson Game

Nim

Nim$^\infty$?

Conclusion

1. Background
2. The Hilbert-Dickson Game
3. Nim
4. Nim$^\infty$?
5. Conclusion

§1. Background

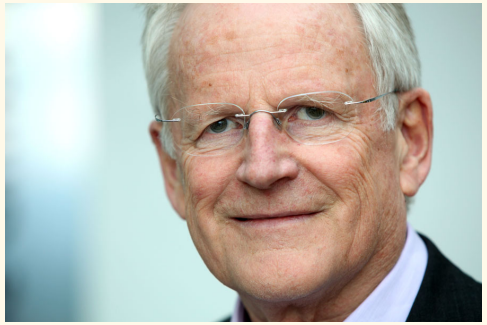
My field, in portraits

Euclid

+

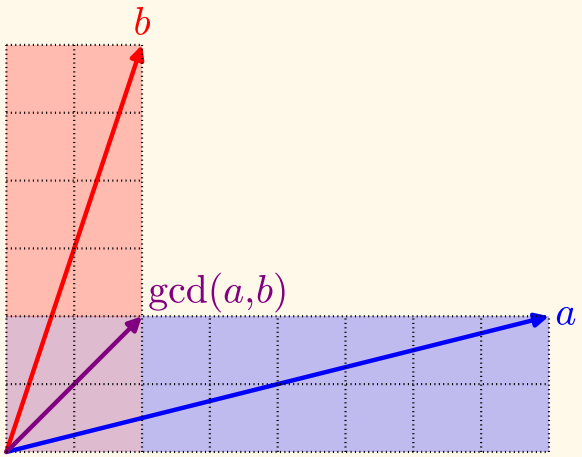
Gauß

=



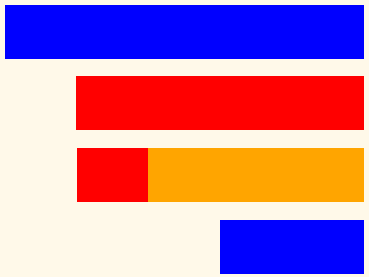
Buchberger

My field, in diagrams



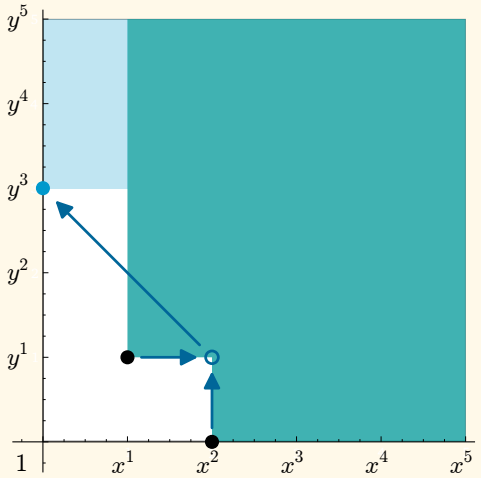
Euclidean algorithm

+



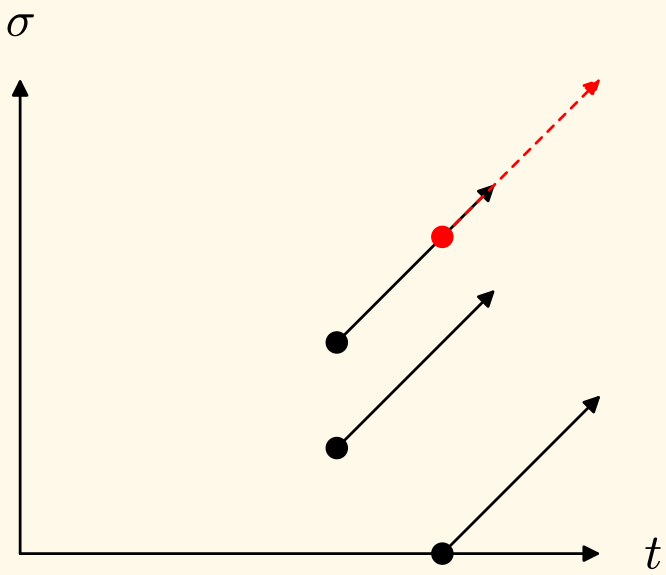
Gauss-Jordan reduction

=



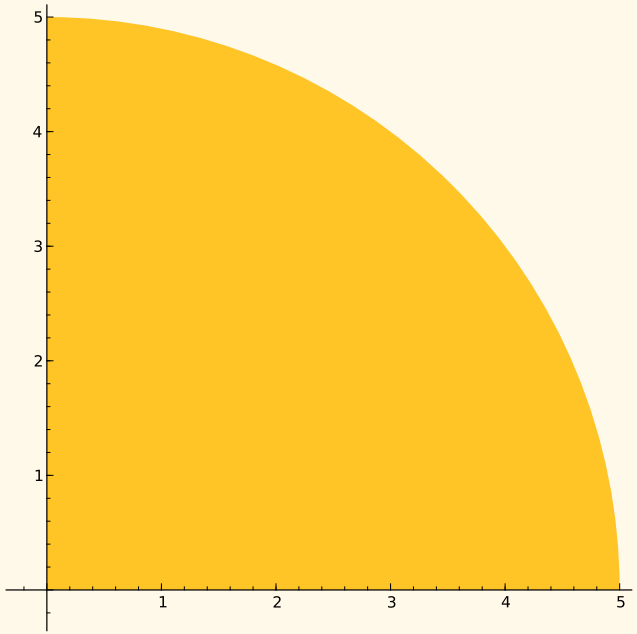
Buchberger's algorithm

Recent contributions



primitive $\mathcal{S}$-irreducible polynomials

(w/Alberto Arri, Google Corp.)

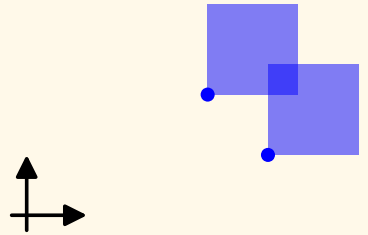


border vectors

(w/Massimo Caboara, Università di Pisa)

Animated commutative algebra

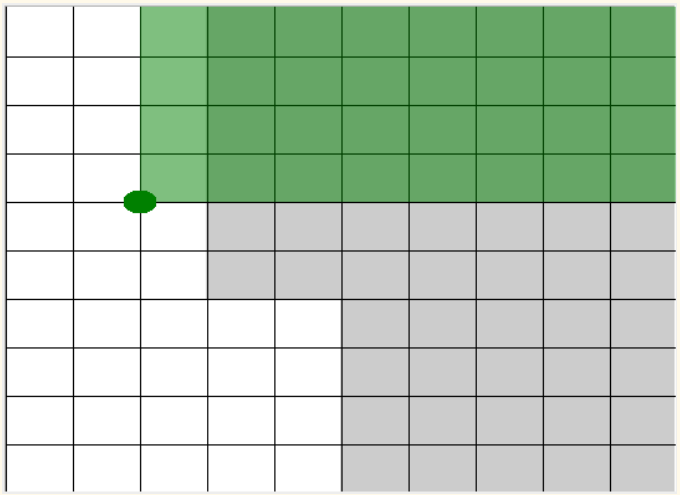
- Ideal
  - generators
  - absorption property
- Quotient ring
- Noetherian
  - Dickson's Lemma
  - Hilbert Basis Theorem
- Hilbert function

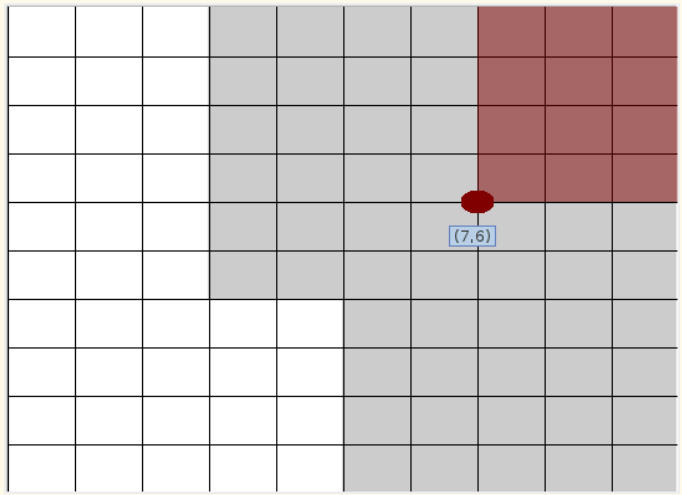


How to communicate this to students?

§2. The Hilbert-Dickson Game

Shall we play a game?

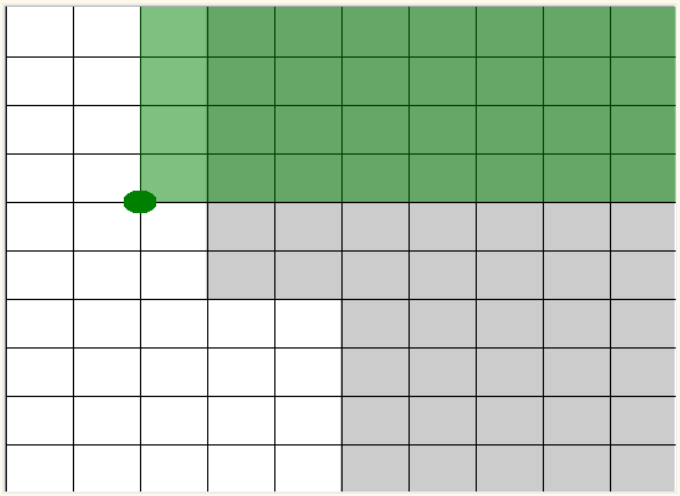


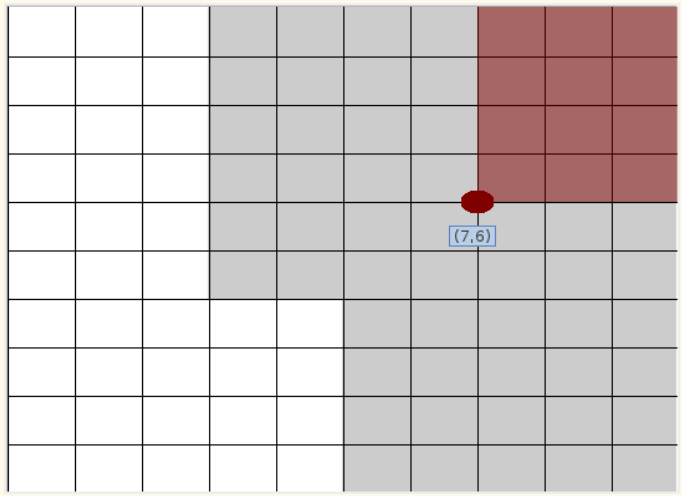


Rules (v. 1)

- Move: choose (x, y)
  - lattice point
  - not northeast of prior move (gray)
- Last move loses

Shall we play a game?





Rules (v. 1)

- Move: choose (x, y)
  - lattice point
  - not northeast of prior move (gray)
- Last move loses

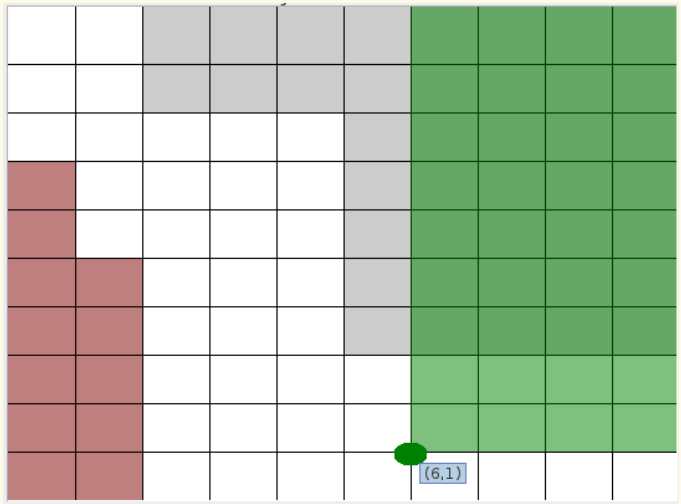
Gameplay

- Too easy: choose $(1, 1)$
- Reflect opponent's choices, force into a corner
- Demonstratio in tabula

Shall we play a game?

Rules (v. 3)

- Choose several lattice points, G
- Move: choose (x, y)
  - lattice point
  - not northeast of prior move (gray)
  - not southwest of point in G (red)
- last move wins



Rules (v. 2)

- Choose several lattice points, G
- For each $d \in \mathbb{N}$, count # of points not southwest of G
  - call this $H(d)$
- Choose lattice point (x, y):
  - not northeast of previously-chosen point
  - For each d, must leave $H(d)$ points southwest of choices

Commutative algebra in action!

Rules?

v. 1 not northeast of previously-chosen point?

- Ascending Chain Condition (Noetherian)
  - Dickson's Lemma / Hilbert Basis Theorem

Commutative algebra in action!

Rules?

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v. 3 not southwest of point in G?

- move in ideal

Commutative algebra in action!

Rules?

v. 1 not northeast of previously-chosen point?

- Ascending Chain Condition (Noetherian)
  - Dickson's Lemma / Hilbert Basis Theorem

v. 3 not southwest of point in G?

- move in ideal

v. 2 leave $H(d)$ points southwest of of choices?

- $H(d)$: Hilbert function
- "invariant" of ideal
- compute basis wrt different ordering

"Hilbert-Dickson" Game

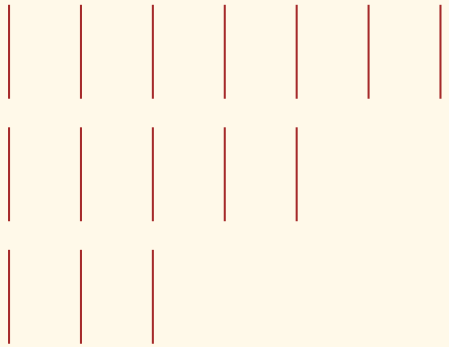
let's try it!

§3. Nim

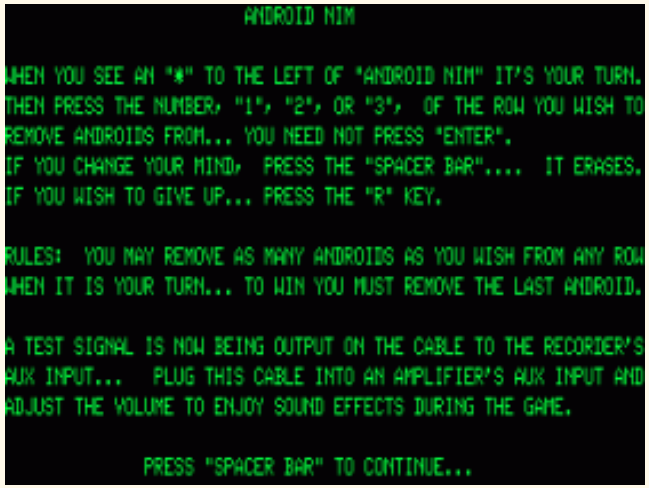
Ancient game

Rules

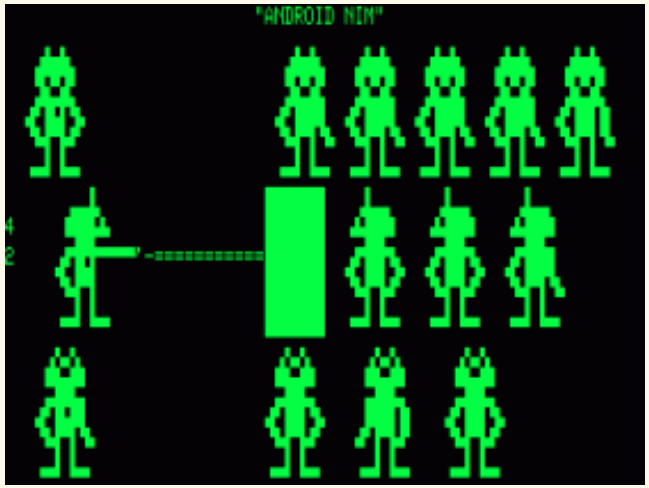
- three rows of sticks
  - usually 7, 5, 3
- can take any number of sticks from one row
- winner takes last sticks



A 10 year-old's introduction to Nim



instructions



high-definition graphics in 1978
(that rectangle is an explosion)

Developer: middle school teacher as aid to teach binary system

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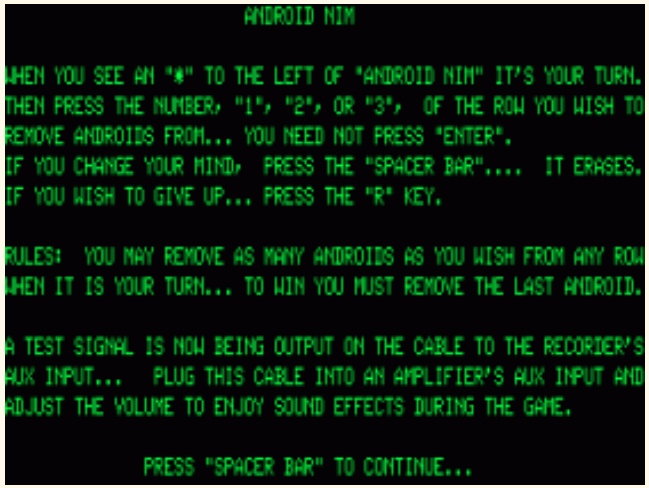

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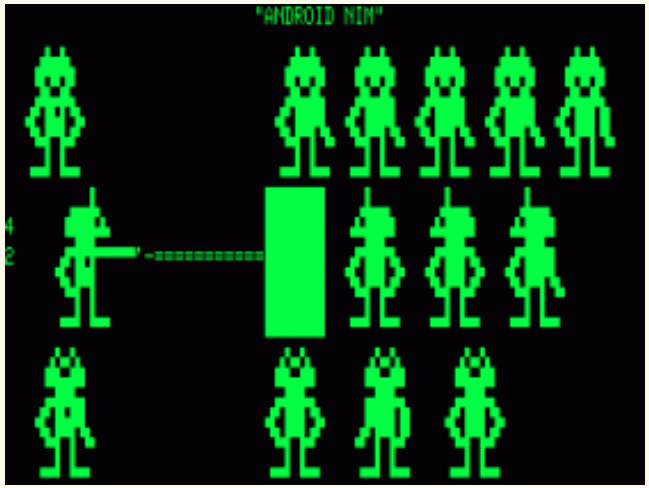

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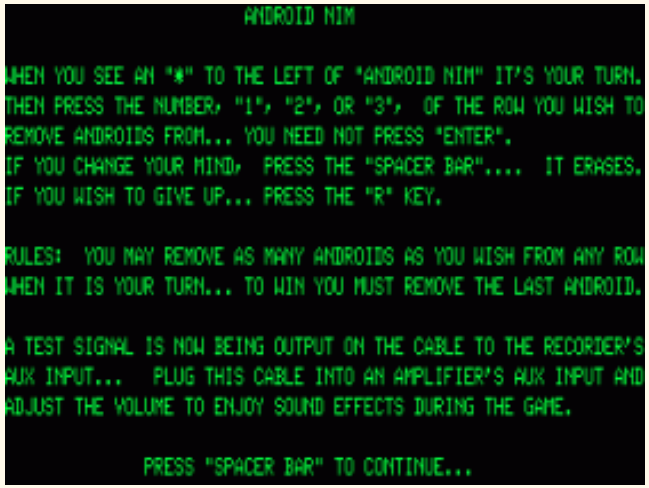

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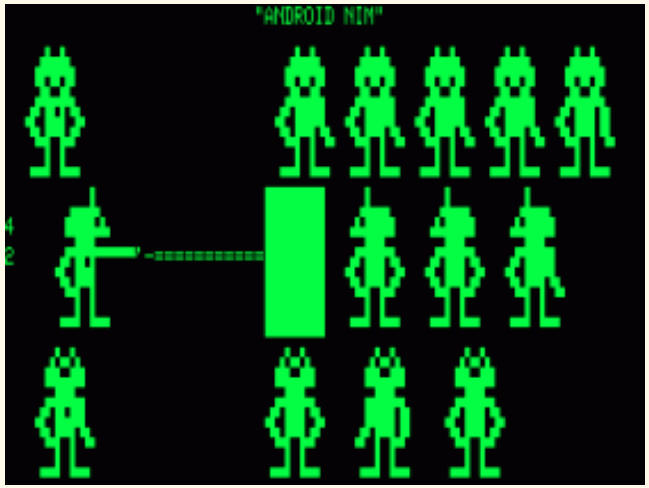

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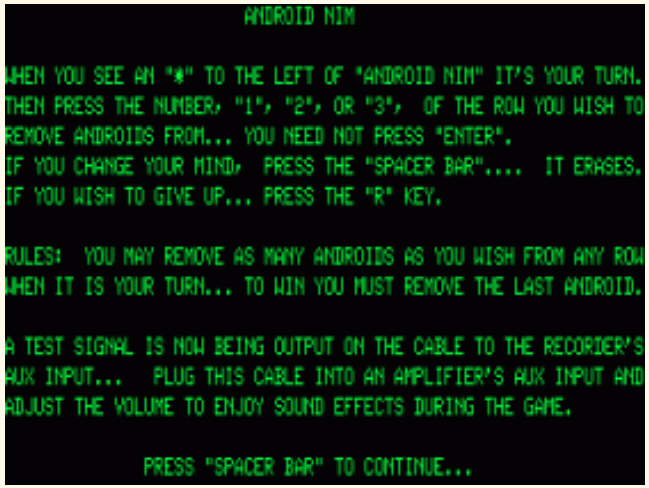

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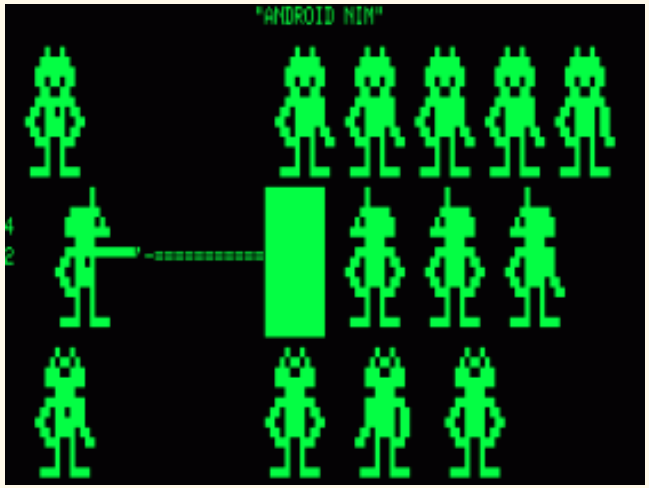

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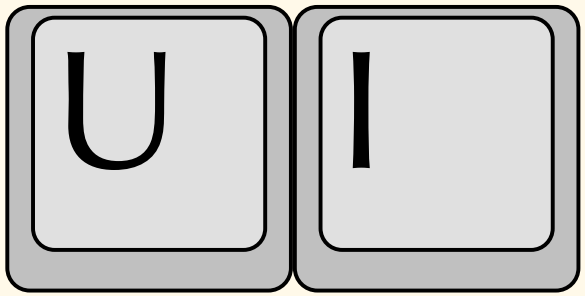
(*Consolation:* no one else won)

Theorem (Bouton)

Let n be the nimber corresponding to the current heaps. If $n = 0$, next player loses.



Theorem (Bouton)

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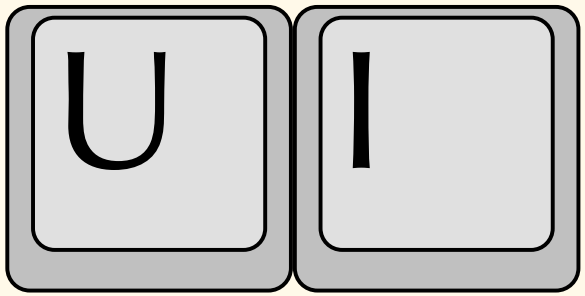
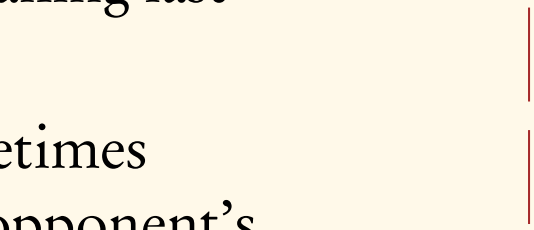
"nimber" is not a typo

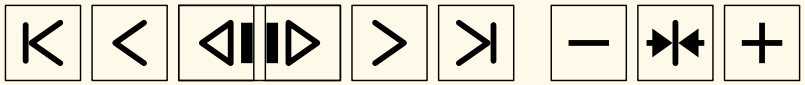


Nimbers: Nim heaps

Basic idea:

- win by taking last sticks
- can sometimes "undo" opponent's "do"
- **goal:** leave "0" sticks
  - modulo some "undo"'s





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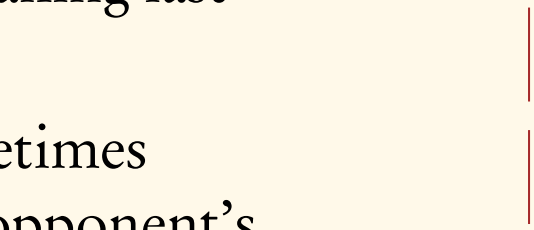
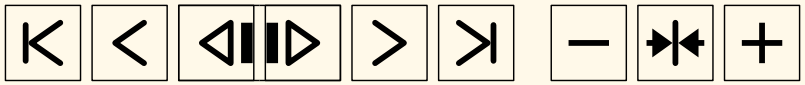
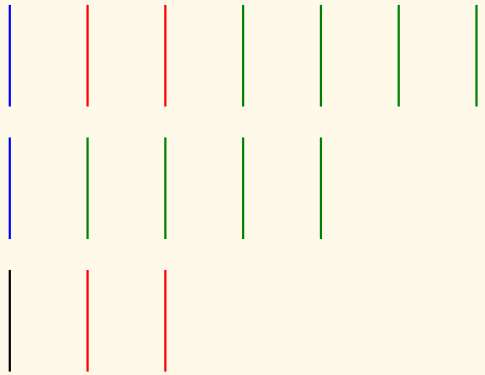




*Solution: base-2 (**xor**) arithmetic*

Examples



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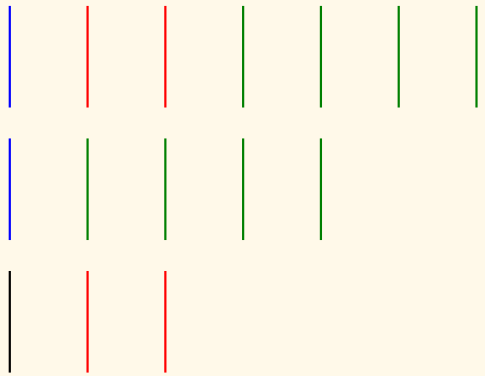
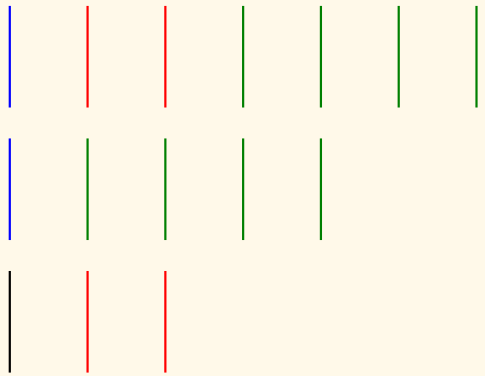


$$(1 \oplus 2) \oplus (1 \oplus 4) \oplus (1 \oplus 2 \oplus 4) = 1$$

to get 0, take 1 piece!

Examples



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Try w/real game

In what sick world is this mathematics?

You may have thought that mathematics was a pretty serious business, and a herd of cows rampaging through a maze, watched by a gang of engineers who are either building the maze or demolishing it, lacks the proper *gravitas*. But, as I've said many times now, 'serious' need not equate to 'solemn'.

— Ian Stewart
Cows in the Maze

O ye of little faith!

- Charles Bouton (Harvard), "Nim: a game with a complete mathematical theory", *Annals of Mathematics*, 1901
- Roland (R. P.) Sprague (Berlin-Charlottenburg, later Freien Universität Berlin), "Über mathematische Kampfspiele", *Tohoku Mathematical Journal*, 1935
- Patrick (P. M.) Grundy (Cambridge, later Oxford), "Mathematics and Games", *Eureka*, 1939.
- John Conway (Princeton), *On Numbers and Games*, 1976
- *Winning Ways for Your Mathematical Plays*, 1982
  - Elwyn Berlekamp (UC Berkeley)
  - John Conway (Princeton)
  - Richard Guy (Erdős number 1)

Sprague-Grundy Theorem

Every "impartial game" is equivalent to a nimber.

Sprague-Grundy Theorem

Every "impartial game" is equivalent to a nimber.

"impartial game"???

Shall we play a game?

game

- two players, alternating turns
- deterministic (no dice)
- transparent information (no cards)
- someone must win in finite time
- **impartial**
  - all moves, rewards available to either player
  - only difference b/w players is who goes first
- **partizan**
  - different players have different choices of move

Examples

Impartial games

- Nim (duh)
- Chompo
- Kayles
- Sprouts
- “poset games”

Partizan games

- Chess
- Go
- Hackenbush

Numbers and games

Background

The Hilbert-Dickson Game

Nim

Nim$^\infty$?

Conclusion

Partizan games $\Longrightarrow$ "Surreal numbers" (Conway, Knuth)

- $\{$# moves after green moves $|$ # moves after blue moves$\}$

Numbers and games

Partizan games $\Longrightarrow$ "Surreal numbers" (Conway, Knuth)

- $\{$# moves after green moves $|$ # moves after blue moves$\}$
- $\{|\} =$ "0"
- new "numbers"? let $a < b$
  - $\{a \mid b\}$ is "simplest" number "between" a, b

    "simplest"? technical details. don't ask.
  - a, b can be lists of numbers
    - $\max(a) \leq \min(b)$

Numbers and games

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Conway: green = Left, blue = Right

"We favor Left"

Examples

$\mathbb{Z}$, of course

- $\{0 \mid\} = 1$
- $\{\mid 0\} = -1$
- $\{1 \mid\} = 2$, $\{2 \mid\} = 3$, etc.

"left is positive"

other powers of 2

- $\{0 \mid 1\} = -\frac{1}{2}$, $\left\{1\frac{1}{4}, 2\right\} = 1\frac{1}{2}$

Examples

$\mathbb{Z}$, of course

- $\{0\,|\}=1$ "left is positive"
- $\{|\,0\}=-1$
- $\{1\,|\}=2$, $\{2\,|\}=3$, etc.

other powers of 2

- $\{0\,|\,1\}=-\frac{1}{2}$, $\left\{1\frac{1}{4},2\right\}=1\frac{1}{2}$

things that make you go, "ωhoah" and "ϵek"

- $\{1,2,\ldots\,|\}=\omega$hoah! — an "infinite" number
- $\left\{|\,-1,-\frac{1}{2},-\frac{1}{4},\ldots\right\}=-\epsilon$ek! a negative "infinitesimal"

Examples

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- $\{|\,0\} = -1$
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But is it useful?

Perfectly useful arithmetic —

— and *largest possible ordered field!*

Nimbers and numbers

Nim impartial $\Longrightarrow$ all moves available to either player

- $\{a \mid a\} = *a$, a "nimber"
  - $*a \oplus *b = \text{mex}(\{c + b : c < a\} \cup \{a + d : d < b\})$

Nimbers and numbers

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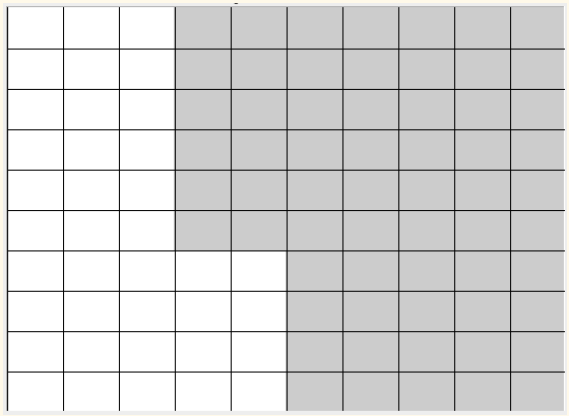
Mex rule

$\{a \mid b\}$ **m**inimal, simplest* **ex**cluded number b/w a, b

*still a technical detail — **don't ask**

§4. Nim$^\infty$?

“Hilbert-Dickson”: impartial game. . .



game?

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- transparent information
- win in finite time

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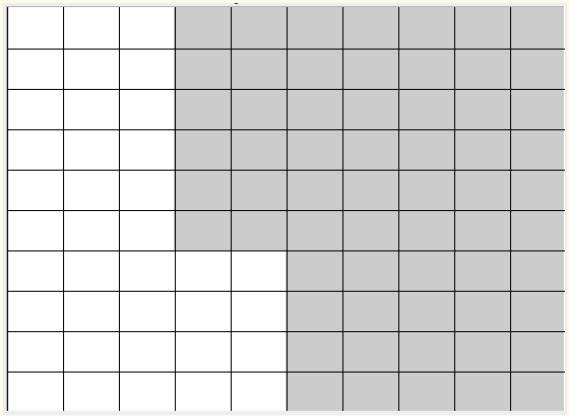
“Hilbert-Dickson”: impartial game. . .



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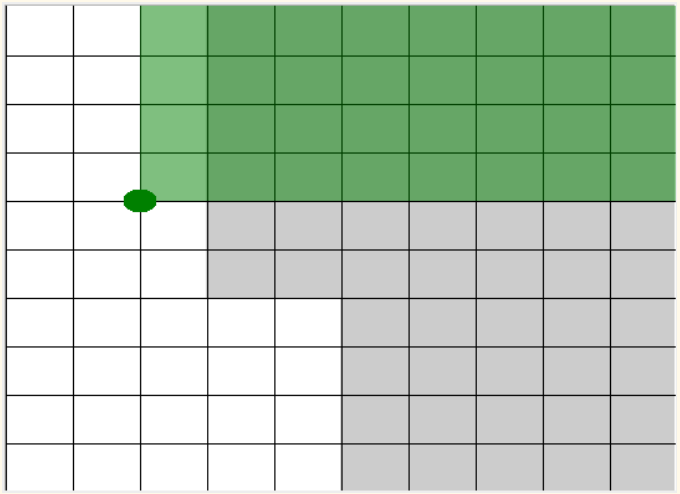
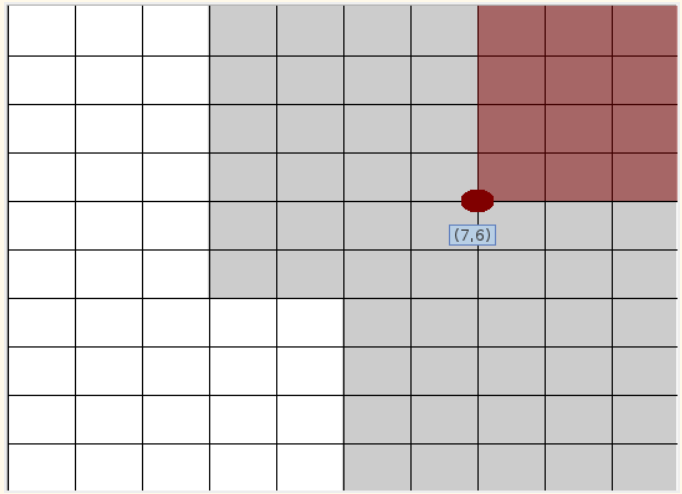
Notice

Sprague-Grundy applies!

. . . variant of Nim. . .

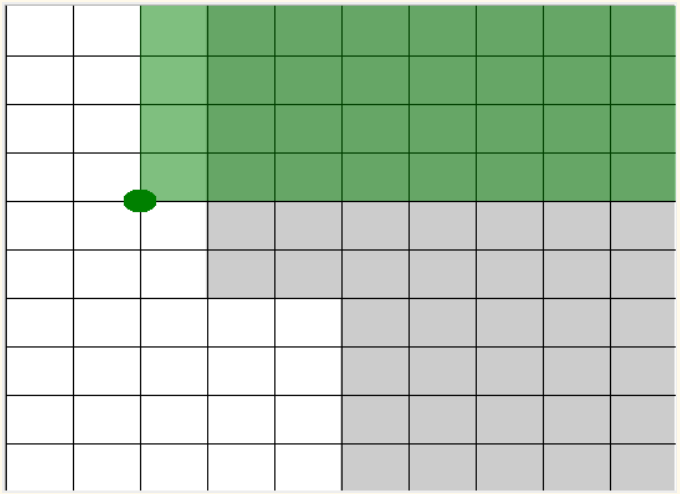


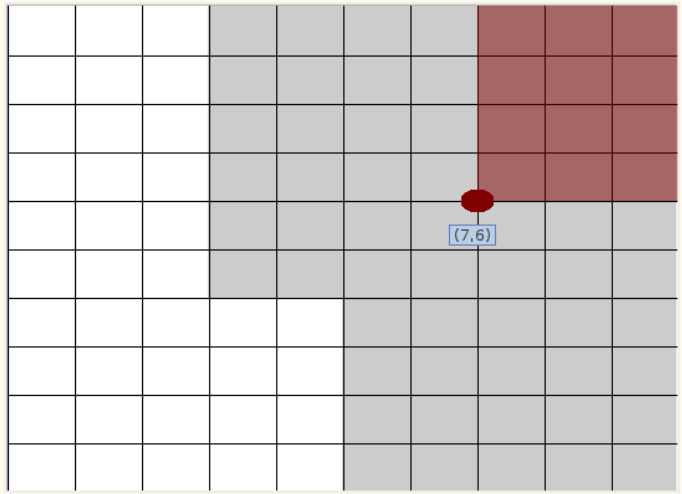


similarities

- rows
- each move removes "sticks"
- analyze w/ nimbers

. . . variant of Nim. . .





similarities

- rows
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differences

- *infinitely* many rows
- affects multiple rows
- ωhoah!
- forbidden positions
  - choices change challenge

. . . known variant?

WWFYMP describes many variants of Nim

- 2d Nim similar to Hilbert-Dickson
  - same 2d board as Hilbert-Dickson
  - n movable pieces
    - move left in same row, *or*
    - down in any column
  - winner moves last piece

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  - n movable pieces
    - move left in same row, *or*
    - down in any column
  - winner moves last piece
- 3d Nim, 4d Nim, . . .
- "Anyone for Hilbert Nim?"

. . . known variant?

WWFYMP describes many variants of Nim

- 2d Nim similar to Hilbert-Dickson
  - same 2d board as Hilbert-Dickson
  - n movable pieces
    - move left in same row, *or*
    - down in any column
  - winner moves last piece
- 3d Nim, 4d Nim, . . .
- "Anyone for Hilbert Nim?"

No variant equivalent to "Hilbert-Dickson game"

Means justify ends!

Nim$^\infty$

Hilbert-Dickson appears to be new

§5. Conclusion

Not bad for a ~~day's~~ few months' work

I get paid to play games!

Not bad for a ~~day's~~ few months' work

I get paid to play games!

er… ahem…

Commutative algebra

- ideals, absorption, Noetherian rings, Dickson's Lemma, Hilbert Basis Theorem, …

Combinatorial game theory

- Nim, nimbers, binary, mex, …

Yet another variant

Gröbner basis in $\mathbb{F}_2[x,y]$

- gameboard? same as Hilbert-Dickson game
- polynomial $\rightsquigarrow$ monomials $\rightsquigarrow$ pieces
  - each polynomials has "distinguished" monomial
- moves?
  - select 2 polynomials $\rightsquigarrow$ shift to cancel disinguished monomials
  - reduce result
- end with "forbidden zone" of Hilbert-Dickson game

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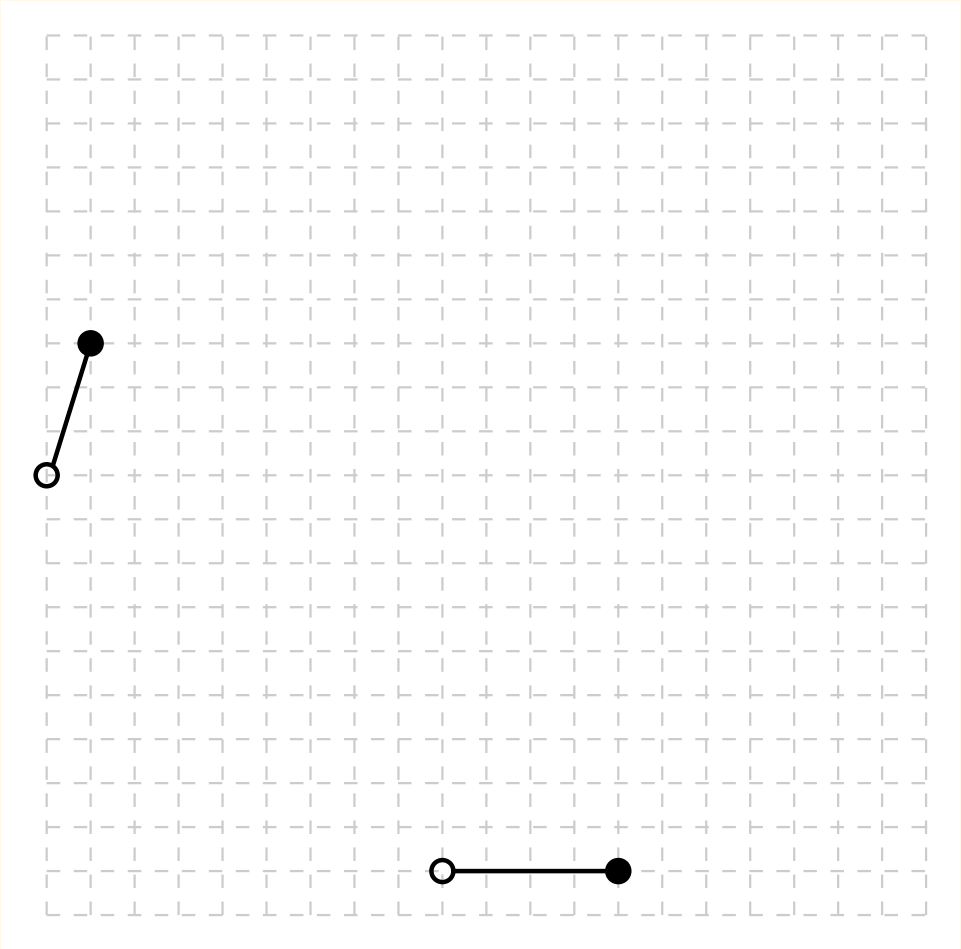
$\therefore$ computing GB is "variant" of "Nim"

An animation is worth 1,000,000 words



Questions

mex: a strategy to compute GB's?

Normal pick *smallest uncomputed* pair

Signature-based pick *smallest excluded* signature

Involutive pick *smallest uncomputed* extension

$\exists$ **Nim-based** strategy?

Thank you

The end

Fine

Finis

конец