#### $\operatorname{Nim}^\infty$

John Perry

Background

The Hilber Dickson

Game

Nim

Nim∞a

Conclusion

# $\operatorname{Nim}^{\infty}$

or, From the ridiculous to the sublime is but a step. (with apologies to Napoleon)

### John Perry

Department of Mathematics University of Southern Mississippi

March 22, 2013

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### Overview

#### $\operatorname{Nim}^\infty$

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### 1 Background

2 The Hilbert-Dickson Game

### 3 Nim





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# §1. Background

# My field, in portraits

### $\operatorname{Nim}^\infty$

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The Hilbert Dickson Game

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Conclusion



Euclid



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Gauß



Buchberger

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# Recent contributions

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Conclusion



primitive S-irreducible polynomials

(w/Alberto Arri, Google Corp.) border vectors

(w/Massimo Caboara, Università di Pisa)

# Animated commutative algebra

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John Perry

#### Background

The Hilbert-Dickson

Nim

Nim<sup>∞</sup>?

Conclusion

• Ideal

• generators

- absorption property
- Quotient ring
- Noetherian
  - Dickson's Lemma
  - Hilbert Basis Theorem
- Hilbert function

How to communicate this to students?

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Background

The Hilbert-Dickson Game

Nim

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 $Nim^{\infty}$ ?

Conclusion

# §2. The Hilbert-Dickson Game

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#### Background



Nim

Nim∞

Conclusion





Rules (v. 1)

- Move: choose (x, y)
  - lattice point
  - not northeast of prior move (gray)
- Last move loses

Nim∞



#### Background



Nim

Nim<sup>∞</sup>?

Conclusion





Rules (v. 1)

- Move: choose (*x*, *y*)
  - lattice point
    not northeast of prior move (gray)
- Last move loses

Gameplay

- Too easy: choose (1, 1)
- Reflect opponent's choices, force into a corner
- Demonstratio in tabula

Rules (v. 3)

- Choose several lattice points, G
- Move: choose (x, y)
  - lattice point
  - not northeast of prior move (gray)
  - not southwest of point in G (red)
- last move wins



### Nim∞

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The Hilbert-Dickson Game

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 $Nim^{\infty}$ ?

Conclusion

Shall we play a game?

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 $\operatorname{Nim}^{\infty}$ 

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#### Background

The Hilbert-Dickson Game

Nim

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Conclusion

### Rules (v. 2)

- Choose several lattice points, G
- For each  $d \in \mathbb{N}$ , count # of points not southwest of G
  - call this H(d)
- Choose lattice point (*x*, *y*):
  - not northeast of previously-chosen point
  - For each *d*, must leave *H*(*d*) points southwest of choices

# Commutative algebra in action!

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#### $\operatorname{Nim}^\infty$

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#### Background

The Hilbert-Dickson Game

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Nim<sup>∞</sup>?

Conclusion

### Rules?

- v. 1 not northeast of previously-chosen point?
  - Ascending Chain Condition (Noetherian)
    - Dickson's Lemma / Hilbert Basis Theorem

# Commutative algebra in action!

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#### Background

The Hilbert-Dickson Game

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Conclusion

### Rules?

- v. 1 not northeast of previously-chosen point?
  - Ascending Chain Condition (Noetherian)
    - Dickson's Lemma / Hilbert Basis Theorem
- v. 3 not southwest of point in G?
  - move in ideal

# Commutative algebra in action!

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#### Background

The Hilbert-Dickson Game

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Nim<sup>∞</sup>?

Conclusion

### Rules?

- v. 1 not northeast of previously-chosen point?
  - Ascending Chain Condition (Noetherian)
    - Dickson's Lemma / Hilbert Basis Theorem
- v. 3 not southwest of point in G?
  - move in ideal
- v. 2 leave H(d) points southwest of of choices?
  - *H*(*d*): Hilbert function
  - "invariant" of ideal
  - compute basis wrt different ordering

### Voilà ici

#### $\mathsf{Nim}^\infty$

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Background

The Hilbert-Dickson Game

Nim

Nim<sup>∞</sup>?

Conclusion

### "Hilbert-Dickson" Game

let's try it!

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Background

The Hilbert Dickson Game

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Conclusion

# §3. Nim

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# Ancient game

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Background

The Hilbert-Dickson Game

Nim

 $Nim^{\infty}$ ?

Conclusion

### Rules

- three rows of sticks
  - usually 7, 5, 3
- can take any number of sticks from one row
- winner takes last sticks

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The Hilbert Dickson Game

Nim

 $\operatorname{Nim}^{\infty}$ ?

Conclusion

#### ANDROID NIM

НЕН YOU SEE WY N® TO THE LEFT OF YAMBODD NUM ITYS YOUR TUBH. НЕН PRESS THE WHOLER. YT., YOU NEED NOT PRESS TENTER. ENVIE ANDROIDS FROM... YOU NEED NOT PRESS TENTER. F YOU CHANGE YOUR HIND, FRESS THE YE NEY. F YOU LENNE OF URU (P.F., FRESS THE YE NEY.

ULES: YOU MAY REMOVE AS MANY ANDROIDS AS YOU WISH FROM ANY ROW MEN IT IS YOUR TURN... TO WIN YOU MUST REMOVE THE LAST ANDROID.

A TEST SIGNAL IS NOW BEING OUTPUT ON THE CABLE TO THE RECORDER'S AUX INPUT... PLUG THIS CABLE INTO AN AMPLIFIER'S AUX INPUT AND ADJUST THE VOLUME TO ENJOY SOUND EFFECTS DURING THE GAME.

PRESS "SPACER BAR" TO CONTINUE...

instructions

high-definition graphics in 1978 (that rectangle is an explosion)

Developer: middle school teacher as aid to teach binary system





 $\operatorname{Nim}^\infty$ 

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#### Background

The Hilbert Dickson Game

Nim

 $Nim^{\infty}$ ?

Conclusion

#### ANDROID NIM

HEN YOU SEE WY \*\* TO THE LEFT OF YAMBODID NUM 175 YOUR TUBH. HEN PRESS THE WAREDR. YT., YOU OR 27, OF THE ROU YOU HISH TO EXVICE ANBOLIDS FROM... YOU NEED NOT PRESS "SHTEP". F YOU CHANGE YOUR NITW, PRESS THE "SPACER BAR"... IT ERRES. F YOU LINST TO GIVE UP... FRESS THE "SPACER BAR"... IT ERRES.

ULES: YOU MAY REMOVE AS MANY ANDROIDS AS YOU WISH FROM ANY ROW MEN IT IS YOUR TURN... TO WIN YOU MUST REMOVE THE LAST ANDROID.

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Developer: middle school teacher as aid to teach binary system

(failed on me, though I did beat the game once)



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#### Background

The Hilbert Dickson Game

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 $Nim^{\infty}$ ?

Conclusion

#### ANDROID NIM

HEN YOU SEE AN "A" TO THE LEFT OF "WARROID NOM" IT'S YOUR TURN. HEN PRESS THE NUMBER. "1," '2", OR "3", OF THE ROU YOU UISH TO ENVICE ANAROIDS FROM.. YOU NEED NOT PRESS "THE "SPACER BAR"..., IT ERWES. F YOU CHANCE YOUR MIND, PRESS THE "SPACER BAR"..., IT ERWES. F YOU WISH TO GIVE UP... PRESS THE "R" KEY.

ULES: YOU MAY REMOVE AS MANY ANDROIDS AS YOU WISH FROM ANY ROW MEN IT IS YOUR TURN... TO WIN YOU MUST REMOVE THE LAST ANDROID.

A TEST SIGNAL IS NOW BEING OUTPUT ON THE CABLE TO THE RECORDER'S AUX INPUT... PLUG THIS CABLE INTO AN AMPLIFIER'S AUX INPUT AND ADJUST THE VOLUME TO ENJOY SOUND EFFECTS DURING THE GAME.

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Developer: middle school teacher as aid to teach binary system

(failed on me, though I did beat the game once)

(in a practice session, so I didn't win a prize)



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Conclusion

#### ANDROID NIM

HEN YOU SEE AN "A" TO THE LEFT OF "WARROID NOM" IT'S YOUR TURN. HEN PRESS THE NUMBER. "1," '2", OR "3", OF THE ROU YOU UISH TO ENVICE ANAROIDS FROM.. YOU NEED NOT PRESS "THE "SPACER BAR"..., IT ERWES. F YOU CHANCE YOUR MIND, PRESS THE "SPACER BAR"..., IT ERWES. F YOU WISH TO GIVE UP... PRESS THE "R" KEY.

ULES: YOU MAY REMOVE AS MANY ANDROIDS AS YOU WISH FROM ANY ROW MEN IT IS YOUR TURN... TO WIN YOU MUST REMOVE THE LAST ANDROID.

A TEST SIGNAL IS NOW BEING OUTPUT ON THE CABLE TO THE RECORDER'S AUX INPUT... PLUG THIS CABLE INTO AN AMPLIFIER'S AUX INPUT AND ADJUST THE VOLUME TO ENJOY SOUND EFFECTS DURING THE GAME.

PRESS "SPACER BAR" TO CONTINUE ...

instructions

high-definition graphics in 1978 (that rectangle is an explosion)

Developer: middle school teacher as aid to teach binary system

(failed on me, though I did beat the game once)

(in a practice session, so I didn't win a prize)

(Consolation: no one else won)



# Mathematical aspects of Nim

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Background

l'he Hilbert-Dickson Game

Nim

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Conclusion

Theorem (Bouton)

Let *n* be the nimber corresponding to the current heaps. If n = 0, next player loses.

# Mathematical aspects of Nim

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Nim<sup>∞</sup>?

Conclusion

Theorem (Bouton)

Let *n* be the nimber corresponding to the current heaps. If n = 0, next player loses.

"nimber" is not a typo

# Nimbers: Nim heaps

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The Hilbert-Dickson Game

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 $Nim^{\infty}$ ?

Conclusion

### Basic idea:

- win by taking last sticks
- can sometimes
   "undo" opponent's
   "do"
- goal: leave "0" sticks
  - modulo some "undo"'s

# Nimbers: Nim heaps

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Background

The Hilbert-Dickson Game

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 $Nim^{\infty}$ ?

Conclusion

### Basic idea:

- win by taking last sticks
- can sometimes
   "undo" opponent's
   "do"
- goal: leave "0" sticks
  - modulo some "undo"'s

Solution: base-2 (xor) arithmetic



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# In what sick world is this mathematics?

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Background

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 $Nim^{\infty}$ ?

Conclusion

You may have thought that mathematics was a pretty serious business, and a herd of cows rampaging through a maze, watched by a gang of engineers who are either building the maze or demolishing it, lacks the proper *gravitas*. But, as I've said many times now, 'serious' need not equate to 'solemn'.

> — Ian Stewart Cows in the Maze

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# O ye of little faith!

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#### Background

The Hilbert-Dickson Game

Nim

Nim<sup>∞</sup>?

Conclusion

- Charles Bouton (Harvard), "Nim: a game with a complete mathematical theory", *Annals of Mathematics*, 1901
- Roland (R. P.) Sprague (Berlin-Charlottenburg, later Freien Universität Berlin), "Über mathematische Kampfspiele", *Tohoku Mathematical Journal*, 1935
- Patrick (P. M.) Grundy (Cambridge, later Oxford), "Mathematics and Games", *Eureka*, 1939.
- John Conway (Princeton), On Numbers and Games, 1976
- Winning Ways for Your Mathematical Plays, 1982
  - Elwyn Berlekamp (UC Berkeley)
  - John Conway (Princeton)
  - Richard Guy (Erdős number 1)

# More mathematical aspects of Nim

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Background

The Hilbert-Dickson Game

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Conclusion

### Sprague-Grundy Theorem

Every "impartial game" is equivalent to a nimber.

# More mathematical aspects of Nim

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Conclusion

### Sprague-Grundy Theorem

Every "impartial game" is equivalent to a nimber.

"impartial game"???



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Background

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Conclusion

### game

- two players, alternating turns
- deterministic (no dice)
- transparent information (no cards)
- someone must win in finite time
- impartial
  - all moves, rewards available to either player
  - only difference b/w players is who goes first
- partizan
  - different players have different choices of move

## Examples

#### $\operatorname{Nim}^\infty$

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The Hilbert-Dickson Game

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Conclusion

### Impartial games

- Nim (duh)
- Chompo
- Kayles
- Sprouts
- "poset games"

### Partizan games

- Chess
- Go
- Hackenbush

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# Numbers and games

(Conway, Knuth)

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Conclusion

### Partizan games $\implies$ "Surreal numbers"

• {# moves after green moves | # moves after blue moves}

Numbers and games

(Conway, Knuth)

John Perry

Background

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Conclusion

Partizan games  $\implies$  "Surreal numbers"

- {# moves after green moves | # moves after blue moves}
- {|} = "0"
- new "numbers"? let *a* < *b* 
  - {a | b} is "simplest" number "between" a, b
    "simplest"? technical details. don't ask.
  - *a*, *b* can be lists of numbers
    - $\max(a) \le \min(b)$

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Background

l'he Hilbert-Dickson Game

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Conclusion

Partizan games  $\implies$  "Surreal numbers"

- {# moves after green moves | # moves after blue moves}
- {|} = "0"
- new "numbers"? let *a* < *b* 
  - {a | b} is "simplest" number "between" a, b
    "simplest"? technical details. don't ask.
  - *a*, *b* can be lists of numbers
    - $\max(a) \le \min(b)$

Conway: green = Left, blue = Right

"We favor Left"

うせん 同・ 本田・ 本田・ 本国・ そう

Numbers and games

(Conway, Knuth)

# Examples

"left is positive"

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'he Hilbert-Dickson Game

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 $Nim^{\infty}$ ?

Conclusion

# $\mathbb{Z}$ , of course

- {0|} = 1
- {| 0} = −1

• 
$$\{1 \mid\} = 2, \{2 \mid\} = 3, \text{ etc.}$$

other powers of 2

• 
$$\{0 \mid 1\} = -\frac{1}{2}, \{1\frac{1}{4}, 2\} = 1\frac{1}{2}$$

## Examples

"left is positive"

#### $\operatorname{Nim}^\infty$

John Perry

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Conclusion

 $\mathbb{Z}$ , of course

- {0|} = 1
- {| 0} = −1

• 
$$\{1 \mid\} = 2, \{2 \mid\} = 3, \text{ etc.}$$

other powers of 2

•  $\{0 \mid 1\} = -\frac{1}{2}, \{1\frac{1}{4}, 2\} = 1\frac{1}{2}$ 

things that make you go, " $\omega$ hoah" and " $\epsilon$ ek"

- $\{1, 2, \dots |\} = \omega$ hoah! an "infinite" number
- $\left\{ |-1, -\frac{1}{2}, -\frac{1}{4}, \ldots \right\} = -\epsilon \text{ek! a negative "infinitesimal"}$

きょうかい 御子 (山下) (日本)

	Examples	$Nim^{\infty}$
	1	John Perry
7 of course		Background
• $\{0 \mid\} = 1$	"left is positive"	The Hilbert- Dickson Game
• $\{ 0\} = -1$		Nim
• $\{1 \mid\} = 2, \{2 \mid\} = 3, \text{ etc.}$		$Nim^{\infty}$ ?
other powers of 2		Conclusion
• $\{0 \mid 1\} = -\frac{1}{2}, \{1\frac{1}{4}, 2\} = 1\frac{1}{2}$		
things that make you go, " $\omega$ hoah" and " $\epsilon$ ek"		
• $\{1, 2, \dots  \} = \omega$ hoah! — an "infinite" number		
• $\left\{ \left  -1, -\frac{1}{2}, -\frac{1}{4}, \ldots \right\} = -\epsilon \text{ek! a negative "infinitesimal"} \right\}$		
But is it useful?		
Perfectly useful arithmetic —		

- and largest possible ordered field!

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# Nimbers and numbers

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Nim impartial  $\implies$  all moves available to either player

- $\{a \mid a\} = *a, a$  "nimber"
  - $*a \oplus *b = \max(\{c+b: c < a\} \cup \{a+d: d < b\})$

 $Nim^{\infty}$ John Perry

Background

The Hilbert Dickson Game

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Conclusion

# Nimbers and numbers

Nim impartial  $\implies$  all moves available to either player

• 
$$\{a \mid a\} = *a, a$$
 "nimber"

•  $*a \oplus *b = \max(\{c+b: c < a\} \cup \{a+d: d < b\})$ 

### Mex rule

 $\{a \mid b\}$  minimal, simplest\* excluded number b/w a, b

\*still a technical detail – don't ask

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Nim∞ John Perry

Background

The Hilbert Dickson Game

Nim

 $Nim^{\infty}$ ?

Conclusion

 $\operatorname{Nim}^\infty$ 

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# $4. \operatorname{Nim}^{\infty}?$

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# "Hilbert-Dickson": impartial game...



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### game?

### impartial?

- two alternating players
- deterministic
- transparent information
- win in finite time

- all moves, rewards available to either player
- only difference b/w players is who goes first

# "Hilbert-Dickson": impartial game...



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### game?

### impartial?

- two alternating players
- deterministic
- transparent information
- win in finite time

- all moves, rewards available to either player
- only difference b/w players is who goes first

### Notice

Sprague-Grundy applies!

# ... variant of Nim...



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## similarities

- rows
- each move removes "sticks"
- analyze w/ nimbers

# ... variant of Nim...





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## similarities

- rows
- each move removes "sticks"
- analyze w/ nimbers

# differences

- *infinitely* many rows
- affects multiple rows
- ωhoah!
- forbidden positions
  - choices change challenge

# ... known variant?

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WWFYMP describes many variants of Nim

- 2d Nim similar to Hilbert-Dickson
  - same 2d board as Hilbert-Dickson
  - *n* movable pieces
    - move left in same row, or
    - down in any column
  - winner moves last piece

# ... known variant?

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- 3d Nim, 4d Nim, ...
- "Anyone for Hilbert Nim?"

# ... known variant?

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- "Anyone for Hilbert Nim?"

No variant equivalent to "Hilbert-Dickson game"

# Means justify ends!

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### $\operatorname{Nim}^\infty$

Hilbert-Dickson appears to be new

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# §5. Conclusion

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# Not bad for a day's few months' work

I get paid to play games!



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# Not bad for a day's few months' work

### I get paid to play games!



er... ahem...

Commutative algebra

- ideals, absorption, Noetherian rings, Dickson's Lemma, Hilbert Basis Theorem, ...
- Combinatorial game theory
  - Nim, nimbers, binary, mex, ...

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# Yet another variant

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Gröbner basis in  $\mathbb{F}_2[x, y]$ 

- gameboard? same as Hilbert-Dickson game
- polynomial --> monomials --> pieces
  - each polynomials has "distinguished" monomial
- moves?

  - reduce result
- end with "forbidden zone" of Hilbert-Dickson game

# Yet another variant

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Gröbner basis in  $\mathbb{F}_2[x, y]$ 

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  - each polynomials has "distinguished" monomial
- moves?

  - reduce result
- end with "forbidden zone" of Hilbert-Dickson game

∴ computing GB is "variant" of "Nim"

# An animation is worth 1,000,000 words

### $\mathrm{Nim}^\infty$

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### Questions

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mex: a strategy to compute GB's?

Normal pick *smallest uncomputed* pair Signature-based pick *smallest excluded* signature Involutive pick *smallest uncomputed* extension

∃ Nim-based strategy?

# Thank you

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Conclusion

# The end

Fine

Finis

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