Determinants in Wonderland

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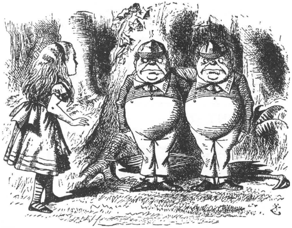

MAA Local Meeting, Pensacola FL

Outline

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Goal

- $\det A = ?$
- Common approaches
  - Patterns
    - 2×2
    - 3×3
  - Expansion by cofactors
  - Triangulation/Gaussian elimination

Goal

- $\det A = ?$
- Common approaches
  - Patterns
    - 2×2

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 3 \cdot 2$$

    - 3×3
  - Expansion by cofactors
  - Triangulation/Gaussian elimination

Goal

- $\det A = ?$
- Common approaches
  - Patterns
    - 2×2
    - 3×3

$$\begin{vmatrix} -1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (-1)\cdot 5\cdot 9 + 2\cdot 6\cdot 7 + 3\cdot 4\cdot 8$$

$$-7\cdot 5\cdot 3 - 8\cdot 6\cdot(-1) - 9\cdot 4\cdot 2$$

  - Expansion by cofactors
  - Triangulation/Gaussian elimination

Goal

- $\det A = ?$
- Common approaches
  - Patterns
    - 2×2
    - 3×3
  - Expansion by cofactors

$$\begin{vmatrix} -1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (-1)^{1+1} \cdot (-1) \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + (-1)^{1+2} \cdot 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + (-1)^{1+3} \cdot 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

  - Triangulation/Gaussian elimination

Goal

- $\det A = ?$
- Common approaches
  - Patterns
    - 2×2
    - 3×3
  - Expansion by cofactors
  - Triangulation/Gaussian elimination

$$\begin{vmatrix} -1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 3 \\ 0 & 13 & 18 \\ 0 & 22 & 30 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 3 \\ 0 & 13 & 18 \\ 0 & 0 & -\frac{6}{13} \end{vmatrix} = -1 \cdot 13 \cdot \left(-\frac{6}{13}\right) = 6.$$

Drawbacks

- Patterns do not generalize:

$$\begin{vmatrix} -1 & 2 & 3 & 4 \\ 5 & 6 & 9 & 8 \\ 1 & 2 & 3 & 4 \\ -3 & 7 & -2 & 1 \end{vmatrix} = ?$$

- Expansion by cofactors: painful, error-prone, tedious
- Triangulation: not always easy; intermediate fractions

Outline

Another way

Rev. Charles Dodgson

- AKA Lewis Caroll (*Alice in Wonderland*, *Through the Looking Glass*, *The Hunting of the Snark*)
- Mathematician, Oxford trained!
- Devised another method for evaluating determinants [1]
- Revisited in recent *CMJ* article [4]

Dodgson's Method.

- "Condensation": 2×2

$$M_3 = \begin{pmatrix} -1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

- For $i < n - 1$, divide elements of M_i by interior of M_{i-2}

Dodgson's Method.

- "Condensation": 2×2

$$M_3 = \begin{pmatrix} -1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} \begin{vmatrix} -1 & 2 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \\ \\ \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} & \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -13 & -3 \\ -3 & -3 \end{pmatrix}$$

- For $i < n - 1$, divide elements of M_i by interior of M_{i-2}

Dodgson's Method.

- "Condensation": 2×2
- For $i < n-1$, divide elements of M_i by interior of M_{i-2}

$$M_3 = \begin{pmatrix} -1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

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$$M_1 = \frac{30}{5} = 6$$

Dodgson's Method.

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$$M_2 = \begin{pmatrix} -13 & -3 \\ -3 & -3 \end{pmatrix}$$

$$M_1 = \frac{30}{5} = 6$$

Nice and easy!

Problem with Dodgson's Method

- Dividing elements $\Longrightarrow$ division by zero?

$$M_4 = \begin{pmatrix} -1 & 2 & 2 & 4 \\ 5 & 6 & 9 & 8 \\ 1 & 2 & 3 & 4 \\ -3 & 7 & -2 & 1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} -16 & 6 & -12 \\ 4 & {\color{red}0} & 12 \\ 13 & -25 & 11 \end{pmatrix}$$

Partial fix

- Swap rows or columns…

$$M_4 = \begin{pmatrix} -1 & 2 & 2 & 4 \\ 5 & 6 & 9 & 8 \\ -3 & 7 & -2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} -16 & 6 & -12 \\ 53 & -75 & 25 \\ -13 & 25 & -11 \end{pmatrix}$$

…no more zero!

- Drawbacks
  - Lose a lot of information
  - Which rows, columns to swap?

Partial fix

- Swap rows or columns…

$$M_4 = \begin{pmatrix} -1 & 2 & 2 & 4 \\ 5 & 6 & 9 & 8 \\ -3 & 7 & -2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} -16 & 6 & -12 \\ 53 & -75 & 25 \\ -13 & 25 & -11 \end{pmatrix}$$

…no more zero!

- Drawbacks
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Outline

Jacobi's Theorem

Theorem (Jacobi, 1833)

Let

- *M be an $n \times n$ matrix;*
- *A be an $m \times m$ minor of M, where $m < n$;*
- *A' be the corresponding minor of the adjugate of M; and*
- *A^* the complementary $(n-m) \times (n-m)$ minor of M.*

Then

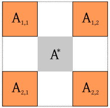
$$\det A' = (\det M)^{m-1} \cdot \det A^* \qquad \text{or} \qquad (\det M)^{m-1} = \frac{\det A'}{\det A^*}.$$

How does this give us Dodgson's method?

Outline

From Jacobi to Dodgson

- 3×3 matrix M
- 1×1 complementary minor A^* (interior!)
- 2×2 minor A



M

$$\det M = \frac{\det A'}{\det A^*}$$

- adjugate from condensation

From Jacobi to Dodgson

- 3×3 matrix M
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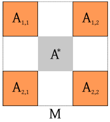
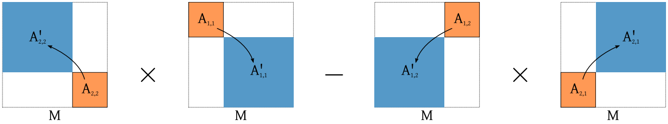


$$\det M = \frac{\det A'}{\det A^*}$$

- adjugate from condensation



Larger matrices?

Theorem

After k successful iterations, Dodgson's Method produces the matrix

$$\begin{pmatrix} |M_{1\ldots k+1,1\ldots k+1}| & |M_{1\ldots k+1,2\ldots k+2}| & \cdots & |M_{1\ldots k+1,n-k\ldots n}| \\ |M_{2\ldots k+2,1\ldots k+1}| & |M_{2\ldots k+2,2\ldots k+2}| & \cdots & |M_{2\ldots k+2,n-k\ldots n}| \\ \vdots & \vdots & \ddots & \vdots \\ |M_{n-k\ldots n,1\ldots k+1}| & |M_{n-k\ldots n,2\ldots k+2}| & \cdots & |M_{n-k\ldots n,n-k\ldots n}| \end{pmatrix};$$

that is, a matrix whose entries are the determinants of all $(k+1)\times(k+1)$ submatrices of M.

By induction...

- First condensation: 2×2 submatrices

$$M_4 = \begin{pmatrix} -1 & 2 & 2 & 4 \\ 5 & 6 & 9 & 8 \\ 1 & 2 & 3 & 4 \\ -3 & 7 & -2 & 1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} \begin{vmatrix} -1 & 2 \\ 5 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 6 & 9 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 9 & 8 \end{vmatrix} \\ \begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 6 & 9 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 9 & 8 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -3 & 7 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 7 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ -2 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -16 & 6 & -12 \\ 4 & 0 & 12 \\ 13 & -25 & 11 \end{pmatrix}$$

- Second condensation: 3×3 submatrices
- Third condensation: 4×4 submatrices

By induction...

- First condensation: 2×2 submatrices
- Second condensation: 3×3 submatrices

$$M_4 = \begin{pmatrix} -1 & 2 & 2 & 4 \\ 5 & 6 & 9 & 8 \\ 1 & 2 & 3 & 4 \\ -3 & 7 & -2 & 1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} -16 & 6 & -12 \\ 4 & 0 & 12 \\ 13 & -25 & 11 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} \begin{vmatrix} -1 & 2 & 2 \\ 5 & 6 & 9 \\ 1 & 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 & 4 \\ 6 & 9 & 8 \\ 2 & 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 5 & 6 & 9 \\ 1 & 2 & 3 \\ -3 & 7 & -2 \end{vmatrix} & \begin{vmatrix} 6 & 9 & 8 \\ 2 & 3 & 4 \\ 7 & -2 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -4 & 8 \\ -50 & -100 \end{pmatrix}$$

- Third condensation: 4×4 submatrices

By induction...

- First condensation: 2×2 submatrices
- Second condensation: 3×3 submatrices
- Third condensation: 4×4 submatrices

$$M_4 = \begin{pmatrix} -1 & 2 & 2 & 4 \\ 5 & 6 & 9 & 8 \\ 1 & 2 & 3 & 4 \\ -3 & 7 & -2 & 1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} -16 & 6 & -12 \\ 4 & 0 & 12 \\ 13 & -25 & 11 \end{pmatrix}$$

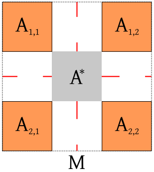
$$M_2 = \begin{pmatrix} -4 & 8 \\ -50 & -100 \end{pmatrix}$$

$$M_1 = \begin{vmatrix} -1 & 2 & 2 & 4 \\ 5 & 6 & 9 & 8 \\ 1 & 2 & 3 & 4 \\ -3 & 7 & -2 & 1 \end{vmatrix}$$

Outline

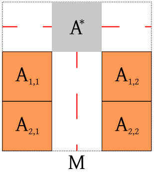
An avenue out?

$A^* = 0$



$\rightsquigarrow$

different A^*

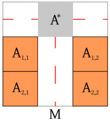


$\Downarrow$

different minor

Choosing a different minor

- 3×3 matrix M, $M_{2,2} = 0$
- 1×1 complementary nonzero minor A^* (non-interior!)
- 2×2 minor A



$$\det M = \frac{\det A'}{\det A^*}$$

- adjugate from condensation

Choosing a different minor

- 3×3 matrix M, $M_{2,2} = 0$
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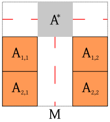

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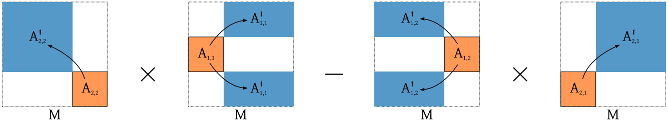

Outline

Easy example

$$M_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

- Dodgson's method fails

Easy example

$$M_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

- Non-zero element above each zero $\rightsquigarrow$ work around

Easy example

$$M_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

- Non-zero element above each zero $\rightsquigarrow$ work around

$$M_3 = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 1 & -1 \\ 4 & -2 & 1 \end{pmatrix}$$

Easy example

$$M_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \qquad M_3 = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 1 & -1 \\ 4 & -2 & 1 \end{pmatrix}$$

- Non-zero element above each zero $\rightsquigarrow$ work around

$$M_2 = \begin{pmatrix} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix}}{1} & \dfrac{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix}}{1} \\ \dfrac{\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix}\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix}}{1} & \dfrac{\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix}}{1} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Easy example

$$M_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \qquad M_3 = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 1 & -1 \\ 4 & -2 & 1 \end{pmatrix}$$

- Non-zero element above each zero $\rightsquigarrow$ work around

$$M_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M_1 = \frac{1}{1}$$

Harder Example

$$M_4 = \begin{pmatrix} -1 & 2 & 2 & 4 \\ 5 & 6 & 9 & 8 \\ 1 & 2 & 3 & 4 \\ -3 & 7 & -2 & 1 \end{pmatrix}$$

Harder Example

$$M_4 = \begin{pmatrix} -1 & 2 & 2 & 4 \\ 5 & 6 & 9 & 8 \\ 1 & 2 & 3 & 4 \\ -3 & 7 & -2 & 1 \end{pmatrix}$$

Encounter 0 at intermediate step:

$$M_3 = \begin{pmatrix} -16 & 6 & -20 \\ 4 & 0 & 12 \\ 13 & -25 & 11 \end{pmatrix}$$

Harder Example

$$M_4 = \begin{pmatrix} -1 & 2 & 2 & 4 \\ 5 & 6 & 9 & 8 \\ 1 & 2 & 3 & 4 \\ -3 & 7 & -2 & 1 \end{pmatrix} \qquad M_3 = \begin{pmatrix} -16 & 6 & -20 \\ 4 & 0 & 12 \\ 13 & -25 & 11 \end{pmatrix}$$

Non-zero above, so choose different 2×2 minor

$$M_2 = \begin{pmatrix} \dfrac{\begin{vmatrix} -16 & 6 \\ 4 & 0 \end{vmatrix}}{6} & \dfrac{\begin{vmatrix} 6 & -20 \\ 0 & 12 \end{vmatrix}}{9} \\ \begin{vmatrix} -1 & 2 & 2 \\ 5 & 6 & 9 \\ -3 & 7 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 2 & 4 \\ 6 & 9 & 8 \\ 7 & -2 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 8 \\ \begin{vmatrix} -1 & 2 & 2 \\ 5 & 6 & 9 \\ -3 & 7 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 2 & 4 \\ 6 & 9 & 8 \\ 7 & -2 & 1 \end{vmatrix} \end{pmatrix}$$

Harder Example

$$M_4 = \begin{pmatrix} -1 & 2 & 2 & 4 \\ 5 & 6 & 9 & 8 \\ 1 & 2 & 3 & 4 \\ -3 & 7 & -2 & 1 \end{pmatrix} \qquad M_3 = \begin{pmatrix} -16 & 6 & -20 \\ 4 & 0 & 12 \\ 13 & -25 & 11 \end{pmatrix}$$

Non-zero above, so choose different 3×3 minor

$$M_2 = \begin{pmatrix} \overset{-4}{\begin{vmatrix} -1 & 2 & 2 \\ 5 & 6 & 9 \\ -3 & 7 & -2 \end{vmatrix}} & \overset{8}{\begin{vmatrix} 2 & 2 & 4 \\ 6 & 9 & 8 \\ 7 & -2 & 1 \end{vmatrix}} \end{pmatrix}$$

Condense alternate minors

$$\begin{vmatrix} -1 & 2 & 2 \\ 5 & 6 & 9 \\ -3 & 7 & -2 \end{vmatrix} \to \begin{vmatrix} -16 & 6 \\ 53 & -75 \end{vmatrix} \to \frac{1200 - 318}{6} = 147$$

$$\begin{vmatrix} 2 & 2 & 4 \\ 6 & 9 & 8 \\ 7 & -2 & 1 \end{vmatrix} \to \begin{vmatrix} 6 & -20 \\ -75 & 25 \end{vmatrix} \to \frac{150 - 1500}{9} = -150$$

Harder Example

$$M_4 = \begin{pmatrix} -1 & 2 & 2 & 4 \\ 5 & 6 & 9 & 8 \\ 1 & 2 & 3 & 4 \\ -3 & 7 & -2 & 1 \end{pmatrix} \qquad M_3 = \begin{pmatrix} -16 & 6 & -20 \\ 4 & 0 & 12 \\ 13 & -25 & 11 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} -4 & 8 \\ 147 & -150 \end{pmatrix}$$

So

$$M_1 = \frac{\det M_2}{6} = \frac{\begin{vmatrix} -4 & 8 \\ 147 & -150 \end{vmatrix}}{6} = \frac{600 - 1176}{6} = -96.$$

Pros and Cons

- Pros
  - More matrices!
  - Relatively easy
  - Lots of reuse
- Cons
  - Identifying minors
  - Bad actors remain

The end

Special thanks:

- Eve Torrence
- Deanna Leggett and Ashley Sanders (students)
- Lewis Carroll and the White Rabbit

Some related works I

Charles Lutwidge Dodgson.
Condensation of Determinants, being a new and brief method for computing their arithmetical values.
Proceedings of the Royal Society of London, 15:150–155, 1866.

Carl Gustav Jacob Jacobi.
De binis quibuslibet functionibus homogeneis secundi ordinis per substitutiones lineares in alias binas transformandis, quae solis quadratis variabilium constant; una cum variis theorematis de transformatione et determinatione integralium multiplicium.
Journal für die reine und angewandte Mathematik, 12:1–69, 1833.

Some related works II

Pierre-Simon Laplace.
Recherches sur le calcul intégral et sur le système du monde.
Histoire de l'Académie Royale des Sciences, pages 267–376, 1772.

Adrian Rice and Eve Torrence.
"Shutting up like a telescope": Lewis Carroll's "Curious" Condensation Method for Evaluating Determinants.
The College Math Journal, 38(2):85–95, 2007.