Combinatorial Criteria for S-polynomial reduction

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Notation:

$[p, q]$	$\operatorname{lcm}(p, q)$
(p, q)	$\gcd(p, q)$
$\overline{f}$	$\operatorname{lt}_{>}(f)$
S_{ij}	$\frac{[\overline{f_i}, \overline{f_j}]}{\overline{f_i}} f_i - \frac{[\overline{f_i}, \overline{f_j}]}{\overline{f_j}} f_j$
S_{ij}^{Θ}	$\frac{[\overline{f_i \circ \Theta}, \overline{f_j \circ \Theta}]}{\overline{f_i \circ \Theta}} f_i \circ \Theta - \frac{[\overline{f_i \circ \Theta}, \overline{f_j \circ \Theta}]}{\overline{f_j \circ \Theta}} f_j \circ \Theta$

Problem:

Given $F, \widetilde{F} \subset F$:

using *only* $\overline{F}$ and $\overline{\widetilde{F}}$,

determine $S_{ij} \underset{\widetilde{F}}{\longrightarrow} 0$

Motivation:

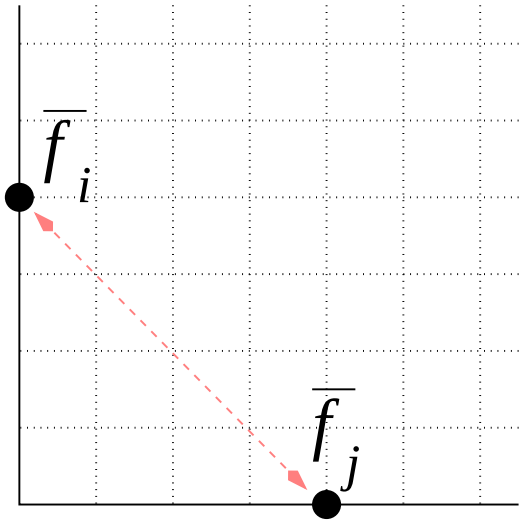
Avoid polynomial division		***speed!***
Exponent structure: $\mathbb{N}$	$\longrightarrow$	***simplicity!***
Unknown criteria?		***curiosity!***

Buchberger's First Criterion: (BC1)

$$\left(\overline{f_i}, \overline{f_j}\right) = 1$$

$$\Downarrow$$

$$S_{ij} \xrightarrow[f_i, f_j]{} 0$$



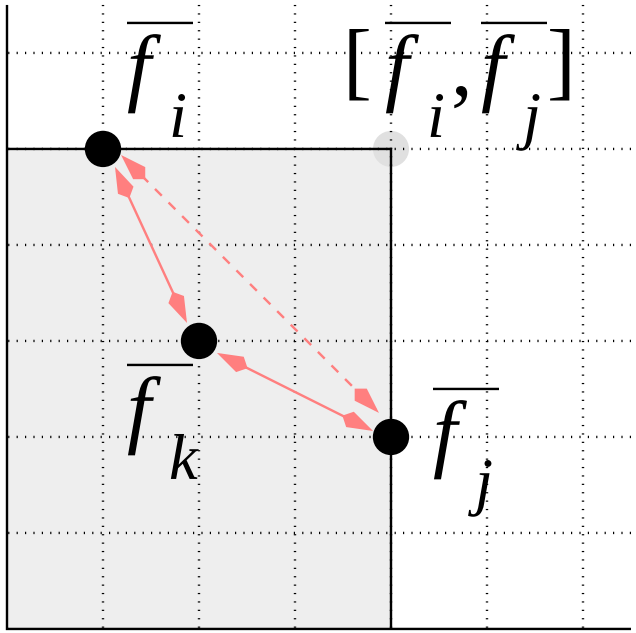
Buchberger's Second Criterion: (BC2)

$$\overline{f_k} \mid [\overline{f_i}, \overline{f_j}]$$

$$S_{ik} \underset{\widetilde{F}}{\rightarrow} 0 \quad S_{jk} \underset{\widetilde{F}}{\rightarrow} 0$$

$$\Downarrow$$

$$S_{ij} \underset{\widetilde{F}}{\rightarrow} 0$$



New Criterion: (NC)

$$\widetilde{\overline{F}} = \{x_0x_1, x_0x_2, x_0x_3\}$$

$$x_i > x_{i+1}$$

$$S_{\widetilde{23}} \underset{\widetilde{\overline{F}}}{\rightarrow} 0 \qquad S_{\widetilde{13}} \underset{\widetilde{\overline{F}}}{\rightarrow} 0$$

$$\Downarrow$$

$$S_{\widetilde{12}} \underset{\widetilde{\overline{F}}}{\rightarrow} 0$$

Syzygy does not suggest this information

Composed BC1: (LCOMM)

$$S_{ij} \to_{\widetilde{F}} 0$$

$$\left[\overline{f_i}, \overline{f_j}\right] \circ \overline{\Theta} = \left[\overline{f_i} \circ \overline{\Theta}, \overline{f_j} \circ \overline{\Theta}\right]$$

$$\Downarrow$$

$$S_{ij}^{\Theta} \xrightarrow[\widetilde{F} \circ \Theta]{} 0$$

Composed BC2:

$$\mathrm{LCOMM}\left(f_i, f_k\right) \wedge \mathrm{LCOMM}\left(f_j, f_k\right)$$

$$\overline{f_k} \circ \overline{\Theta} \mid \left[\overline{f_i} \circ \overline{\Theta}, \overline{f_j} \circ \overline{\Theta}\right]$$

$$\Downarrow$$

$$S_{ij}^{\Theta} \xrightarrow[\widetilde{F} \circ \Theta]{} 0$$

Conjectured Extension of NC:

$$\overline{\widetilde{F}} = \{pq_1, pq_2, pq_3\}$$

$$p > q_1 > q_2 > q_3 \quad (p, q_i) = \left(q_i, q_j\right) = 1$$

$$S_{\widetilde{23}} \underset{\widetilde{F}}{\rightarrow} 0 \quad \wedge \quad S_{\widetilde{13}} \underset{\widetilde{F}}{\rightarrow} 0$$

$$\Downarrow$$

$$S_{\widetilde{12}} \underset{\widetilde{F}}{\rightarrow} 0$$

Summary

Can gain marginal improvements

Some optimizations not suggested by Syz $(\overline{F})$

Future:

"Best possible" result using $\overline{F}$, $\overline{\widetilde{F}}$?

Consider exponent structure